1. Compute

$$\frac{5+\sqrt{6}}{\sqrt{2}+\sqrt{3}}+\frac{7+\sqrt{12}}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{63+\sqrt{992}}{\sqrt{31}+\sqrt{32}}.$$

Answer: $\mathbf{126\sqrt{2}}$

Solution: Rationalizing the denominators turns the numerators into differences of cubes, which gives

$$\begin{aligned}3\sqrt{3}-2\sqrt{2}+4\sqrt{4}-3\sqrt{3}+\cdots+32\sqrt{32}-31\sqrt{31} &= 32\sqrt{32}-2\sqrt{2}\\ &= 128\sqrt{2}-2\sqrt{2}\\ &= \boxed{126\sqrt{2}}.\end{aligned}$$

2. Find the sum of the solution(s) x to the equation

$$x=\sqrt{2022+\sqrt{2022+x}}. \tag{1}$$

Answer: $\mathbf{\frac{1+\sqrt{8089}}{2}}$

Solution: Consider the following equations:

$$y=\sqrt{2022+y} \tag{$*$}$$

$$x=\sqrt{2022+\sqrt{2022+x}} \tag{$**$}$$

Equation $(*)$ and Equation $(**)$ have the same solution, since if you plug the definition into the RHS repeatedly, replacing y for the first equation $\sqrt{2022+y}$ and replacing x in the second equation with $\sqrt{2022+x}$, then you arrive at $x=\sqrt{2022+\sqrt{2022+\sqrt{2022\ldots}}}=y$. To solve for $(*)$, we simply square on both sides and solve the quadratic. We discard the extraneous solution to get $\boxed{\dfrac{1+\sqrt{8089}}{2}}$.

3. Compute $\left\lfloor \frac{1}{\frac{1}{2022}+\frac{1}{2023}+\cdots+\frac{1}{2064}} \right\rfloor$.

Answer: **47**

Solution: Note that

$$\frac{1}{2022}>\frac{1}{2023},\frac{1}{2024},\ldots,\frac{1}{2064}.$$

Therefore,

$$\frac{1}{2022}+\frac{1}{2023}+\cdots+\frac{1}{2064}<43\cdot\frac{1}{2022},$$

which implies that

$$\frac{1}{\frac{1}{2022}+\frac{1}{2023}+\cdots+\frac{1}{2064}}>\frac{1}{43\cdot\frac{1}{2022}}=\frac{2022}{43}=47\frac{1}{43}.$$

Similarly,

$$\frac{1}{2064}<\frac{1}{2022},\frac{1}{2023},\frac{1}{2024},\ldots,\frac{1}{2063}.$$

Therefore,

$$\frac{1}{2022}+\frac{1}{2023}+\cdots+\frac{1}{2064} > 43\cdot\frac{1}{2064}$$

which implies that

$$\frac{1}{\frac{1}{2022}+\frac{1}{2023}+\cdots+\frac{1}{2064}} < \frac{1}{43\cdot\frac{1}{2064}} = \frac{2064}{43} = 48.$$

Therefore, since

$$47\frac{1}{43} < \frac{1}{\frac{1}{2022}+\frac{1}{2023}+\cdots+\frac{1}{2064}} < 48,$$

we have

$$\left\lfloor \frac{1}{\frac{1}{2022}+\frac{1}{2023}+\cdots+\frac{1}{2064}} \right\rfloor = \boxed{47}.$$

4. Let the roots of

$$x^{2022}-7x^{2021}+8x^2+4x+2$$

be $r_1, r_2, \cdots, r_{2022}$, the roots of

$$x^{2022}-8x^{2021}+27x^2+9x+3$$

be $s_1, s_2, \cdots, s_{2022}$, and the roots of

$$x^{2022}-9x^{2021}+64x^2+16x+4$$

be $t_1, t_2, \cdots, t_{2022}$. Compute the value of

$$\sum_{1\le i,j\le 2022} r_is_j + \sum_{1\le i,j\le 2022} s_it_j + \sum_{1\le i,j\le 2022} t_ir_j.$$

Answer: 191

Solution: We wish to compute

$$\sum_{1\le i,j\le 2022} r_is_j + \sum_{1\le i,j\le 2022} s_it_j + \sum_{1\le i,j\le 2022} t_ir_j$$

$$\begin{aligned}
&= \frac{1}{2}((r_1+r_2+\cdots+r_{2022}+s_1+s_2+\cdots+s_{2022}+t_1+t_2+\cdots+t_{2022})^2 \\
&\quad -(r_1^2+r_2^2+\cdots+r_{2022}^2+s_1^2+s_2^2+\cdots+s_{2022}^2+t_1^2+t_2^2+\cdots+t_{2022}^2)) \\
&\quad - \sum_{1\le i<j\le 2022} r_ir_j - \sum_{1\le i<j\le 2022} s_is_j - \sum_{1\le i<j\le 2022} t_it_j.
\end{aligned}$$

We have $r_1^2+r_2^2+\cdots+r_{2022}^2 = (r_1+r_2+\cdots+r_{2022})^2 - 2\sum_{1\le i<j\le 2022} r_ir_j$. Substituting this in for each of r, s, and t gives us

$$\begin{aligned}
&= \frac{1}{2}((r_1+r_2+\cdots+r_{2022}+s_1+s_2+\cdots+s_{2022}+t_1+t_2+\cdots+t_{2022})^2 \\
&\quad -(r_1+r_2+\cdots+r_{2022})^2-(s_1+s_2+\cdots+s_{2022})^2-(t_1+t_2+\cdots+t_{2022})^2).
\end{aligned}$$

Using Vieta's formulas, this is equal to

$$\frac{1}{2}((7+8+9)^2-7^2-8^2-9^2) = 7\cdot 8+8\cdot 9+9\cdot 7 = 191.$$

5. x, y, and z are real numbers such that $xyz = 10$. What is the maximum possible value of $x^3y^3z^3 - 3x^4 - 12y^2 - 12z^4$?

Answer: 760

Solution: We can use the AM-GM inequality to minimize $3x^4+12y^2+12z^4$, which will maximize the overall expression. To make all the exponents the same on the geometric mean side, we split $12y^2$ into $6y^2 + 6y^2$. We have $3x^4 + 6y^2 + 6y^2 + 12z^4 \geq 4\sqrt[4]{1296x^4y^4z^4} = 24xyz = 240$. So, $x^3y^3z^3 - 3x^4 - 12y^2 - 12z^4 \leq 1000 - 240 = \boxed{760}$.

6. Compute

$$\cot\left(\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right).$$

Answer: $\frac{25}{23}$

Solution: Let the sum inside the cot be S. Then, we have

$$S = \sum_{n=1}^{23} \cot^{-1}(1 + n(n+1)).$$

Note that $\cot^{-1} a - \cot^{-1} b = \tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{\frac{1}{a}-\frac{1}{b}}{1+\frac{1}{ab}}\right) = \tan^{-1}\left(\frac{b-a}{ab+1}\right) = \cot^{-1}\left(\frac{ab+1}{b-a}\right)$, so $\cot^{-1}(1 + n(n+1)) = \cot^{-1}\frac{1}{n} - \cot^{-1}\frac{1}{n+1}$. Telescoping, our sum becomes

$$S = \cot^{-1}\frac{1}{1} - \cot^{-1}\frac{1}{24} = \cot^{-1}\frac{25}{23},$$

which gives $\cot S = \boxed{\frac{25}{23}}$.

7. Let $M = \{0, 1, 2, ..., 2022\}$ and let $f : M \times M \to M$ such that for any $a, b \in M$,

$$f(a, f(b, a)) = b$$

and $f(x, x) \neq x$ for each $x \in M$. How many possible functions f are there (mod 1000)?

Answer: 0

Solution: No such functions f exist.

Suppose otherwise. Write $f(b, a) = c$. Then by the condition in the problem, $f(a, c) = b$ and $f(c, b) = f(c, f(a, c)) = a$. Consider the set

$$S = \{(x, y, z) \mid f(x, y) = z,\ x, y, z \in M\}$$

By our observation above, $(x, y, z) \in S$ if and only if $(y, z, x) \in S$. Hence we may partition S into set of the form $\{(x, y, z), (y, z, x), (z, x, y)\}$ for x, y, z not all equal (since it is known $(x, x, x) \notin S$). Hence 3 divides $|S|$. Then, $|S| = |M|^2 = 2023^2$, as to form elements in S, we can arbitrarily choose x and y while z is then determined. This is a contradiction as 3 does not divide 2023^2.

8. For all positive integers $m > 10^{2022}$, determine the maximum number of real solutions $x > 0$ of the equation $mx = \lfloor x^{11/10} \rfloor$.

Answer: 10

Solution: We claim that there can never be more than 10 solutions. Clearly, x will never be very small, since m is so large. Let $x = (k^{10} + \epsilon)$ for some small ϵ and some positive integer k as large as possible. Then we may Taylor Expand

$$k^{11} < m(k^{10} + \epsilon) < (k^{10} + \epsilon)^{11/10} = k^{11} + \frac{11}{10}k\epsilon + \frac{1}{2} \cdot \frac{11}{10} \cdot \frac{1}{10}k^{-9}\epsilon^2 + \cdots$$

It is clear that $\lfloor x^{11/10} \rfloor = \lfloor k^{11} + \frac{11}{10}k\epsilon \rfloor$, since because k is much larger than ϵ, the remaining terms in the summation will be negligible decimals. Using the fact that k is an integer, we may write

$$m(k^{10} + \epsilon) = \left\lfloor (k^{10} + \epsilon)^{11/10} \right\rfloor = \left\lfloor k^{11} + \frac{11}{10}k\epsilon \right\rfloor = k^{11} + \frac{11}{10}k\epsilon - \left\{ \frac{11}{10}k\epsilon \right\}$$

$$(m - k)(k^{10} + \epsilon) = \frac{1}{10}k\epsilon - \left\{ \frac{11}{10}k\epsilon \right\}$$

Now since the LHS is on the order of k^{10}, it must in fact be 0 (we could obtain this by bounding it below my removing the epsilon and getting a contradicting inequality), and so

$$\frac{1}{10}k\epsilon = \left\{ \frac{11}{10}k\epsilon \right\}$$

Therefore, $\frac{11}{10}k\epsilon - \lfloor \frac{11}{10}k\epsilon \rfloor = \frac{1}{10}k\epsilon$ and so $\lfloor \frac{11}{10}k\epsilon \rfloor = k\epsilon$, and so $k\epsilon$ is an integer. Taking the floor of the expression above, we find that $\lfloor \frac{1}{10}k\epsilon \rfloor = 0$. Therefore, $k\epsilon$ can only take on the values $0, 1, 2, \ldots, 9$, and we have achieved $\boxed{10}$ solutions, as desired.

We can additionally show this is achievable. It is not hard to see that $x = (10^{2022}+1)^{10} + \frac{r}{10^{2022}+1}$ for $r = 0, 1, 2, \ldots, 9$ all satisfy this equation for $m = 10^{2022} + 1$.

9. Let $P(x) = 8x^3 + ax^2 + bx + 1$ for $a, b \in \mathbb{Z}$. It is known that P has a root $x_0 = p + \sqrt{q} + \sqrt[3]{r}$, where $p, q, r \in \mathbb{Q}, q \geq 0$; however, P has no *rational* roots. Find the smallest possible value of $a + b$.

Answer: −6

Solution: We have $P(x_0) = 0$ and $x_0 = p + \sqrt{q} + \sqrt[3]{r}$. Note that $P \in \mathbb{Q}[x]$ (since $\mathbb{Q}[x] \equiv \mathbb{Z}[x]$) and $\deg P = 3$. Moreover, observe that if $r = 0$, P has at least one rational root, hence $r \neq 0$. Now consider the polynomial

$$Q(x) = (x - p - \sqrt{q})^3 - r.$$

Trivially, $Q(x_0) = 0$, and $Q \in \mathbb{Q}[\sqrt{q}]$, i.e. the coefficients of Q are of the form $\alpha + \beta\sqrt{q}$ for $\alpha, \beta \in \mathbb{Q}$. Since $\deg Q = 3$, we can express P in terms of Q as

$$P(x) = 8Q(x) + R(x)$$

where $R \in \mathbb{Q}[\sqrt{q}], \deg R \leq 2$. Since $P(x_0) = Q(x_0)$, it follows that $R(x_0) = 0$. We consider cases for the degree of $R(x)$:

- Case 1: $\deg R = 2$. If $R \mid P$, then $\frac{P}{R} \in \mathbb{Q}[q]$ and is of degree 1, hence P has a rational zero, which is a contradiction. If $R \nmid P$, we can divide with remainder to get

$$P(x) = c(x - x_r)R(x) + S(x).$$

  We now have that $S \in \mathbb{Q}[\sqrt{q}]$ and $S(x_0) = 0$, hence $x_0 \in \mathbb{Q}[\sqrt{q}]$ ($\deg S = 1$), which is again a contradict.
- Case 2: $\deg R = 1$. Since $R(x_0) = 0$, as above, $x_0 \in \mathbb{Q}[\sqrt{q}]$ ($\deg S = 1$), which is a contradiction.

We therefore have that $\deg R = 0$, and we can consider without loss of generality that $P(x) = 8Q(x)$ (if $R(x) = c$, c can be absorbed by r). Since the coefficients of P are integers, $q = 0$. Expanding Q we therefore get

$$P(x) = 8Q(x) = 8x^3 - 3 \cdot 8x^2p + 3 \cdot 8xp^2 - 8p^3 - 8r$$

where all coefficients are integers and $8p^3 + 8r = -1$. We seek the minimum value of $a + b = 24(p^2 - p)$. The minimum of the function is at $p = \frac{1}{2}$, for which we find $r = -\frac{1}{4}$ and $P(x) = 8x^3 - 12x^2 + 6x + 1 = (2x - 1)^3 + 2$. Finally, $a + b = -6$.

10. Let $f^1(x) = x^3 - 3x$. Let $f^n(x) = f(f^{n-1}(x))$. Let $\mathcal{R}$ be the set of roots of $\frac{f^{2022}(x)}{x}$. If

$$\sum_{r \in \mathcal{R}} \frac{1}{r^2} = \frac{a^b - c}{d}$$

for positive integers a, b, c, d, where b is as large as possible and c and d are relatively prime, find $a + b + c + d$.

Answer: 4072

Solution: Consider the substitution $x = 2\cos t$. From this we obtain that $f^1(2\cos t) = 8\cos^3 t - 6\cos t = 2(4\cos^3 t - 3\cos t) = 2\cos 3t$. Thus $f^{2022}(2\cos t) = 2\cos 3^{2022} t$. Therefore $f^{2022}(x)$ is the 3^{2022}th Dickson Polynomial and has the form

$$f^{2022}(x) = \sum_{k=0}^{\frac{3^{2022}-1}{2}} \frac{3^{2022}}{3^{2022} - k} \binom{3^{2022} - k}{k} (-1)^k x^{3^{2022} - 2k}$$

The roots of the polynomial $f^{2022}(x)/x$ are all of those of $f^{2022}(x)$ except 0 (which we are guaranteed to have from parity).

We are left to determine the coefficient of x and x^3 in $f^{2022}(x)$ and then we can finish by Vieta's. For simplicity, let $3^{2022} = 2\alpha + 1$.

The linear coefficient is

$$c_1 = \frac{2\alpha + 1}{\alpha + 1} \binom{\alpha + 1}{\alpha} (-1)^\alpha = (2\alpha + 1)(-1)^\alpha$$

The coefficient of x^3 is

$$c_3 = \frac{2\alpha + 1}{\alpha + 2} \binom{\alpha + 2}{\alpha - 1} (-1)^{\alpha - 1} = \frac{(2\alpha + 1)(\alpha + 1)(\alpha)(-1)^{\alpha - 1}}{6}$$

By Vieta's, we wish to find

$$-\frac{2c_3}{c_1}$$

so our desired answer is $\frac{(\alpha+1)(\alpha)}{6}$, and because $\alpha = \frac{3^{2022}-1}{2}$ we have

$$\frac{\left(\frac{3^{2022}-1}{2}\right)\left(\frac{3^{2022}+1}{2}\right)}{6} = \frac{3^{4044}-1}{24}$$

which gives us an answer of $3 + 4044 + 1 + 24 = \boxed{4072.}$