

Time limit: 50 minutes.

Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.

No calculators.

1. A standard 6-sided dice is rolled. What is the expected value of the roll given that the value of the result is greater than the expected value of a regular roll?
2. 5 students, all with distinct ages, are randomly seated in a row at the movies. The probability that, from left to right, no three consecutive students are seated in increasing age order is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
3. Consider sequences that consist of 0's and 1's. The probability that a random sequence of length 2021 contains an equal number of occurrences of '01' and '10' is $\frac{m}{n}$, where m, n are positive, relatively prime integers. Find $m + n$.
4. Sofia has a $2 \times 2 \times 2$ wooden cube. She paints each side with a different color and cuts it into 8 unit cubes. Let N be the number of unique ways she can reassemble the unit cubes into a $2 \times 2 \times 2$ cube, given that the painted faces must always be on the outside of the cube. Rotations along any axis of the whole $2 \times 2 \times 2$ cube do not count as distinct.
If N can be written as $a! \cdot p^b$ with p prime, what is $a + b + p$?
5. Alice and Bob play a game in which they take turns rolling a fair die. The winner of a round is the first player to roll a 6. Whoever loses rolls first in the next round. Alice rolls first in round 1. The probability that Alice rolls first in round 4 is $\frac{A}{B}$, where A and B are relatively prime, positive integers. Find $A + B$.
6. A frog starts at $(0, 0)$ and must return to his home at $(13, 7)$. There is a river located along the line $y = x - 6$. At each step, the frog can only move exactly one unit up or one unit to the right along the lattice points of the plane. If the frog cannot cross the river (but is allowed to move to points on the river), the number of paths the frog can take to his home is N . Compute $\frac{N}{120}$.
7. Anne consecutively rolls a 2020-sided dice with faces labeled from 1 to 2020 and keeps track of the running sum of all her previous dice rolls. She stops rolling when her running sum is greater than 2019. Let X and Y be the running sums she is most and least likely to have stopped at, respectively. What is the ratio between the probabilities of stopping at Y to stopping at X ?
8. Oliver and Xavier are playing a game on an n by n grid of squares. Initially, all cells of the grid are unoccupied. A *turn* is defined as Oliver placing an O on any currently unoccupied square then Xavier placing an X on any remaining unoccupied square. If, at the end of such a turn, there exists a row or column where there are at least 3 more O 's than X 's in that given row or column, then Oliver wins. Otherwise, if they fill the board and there does not exist such a row or column, Xavier wins. Find the minimum value of n such that Oliver is guaranteed to win if they both play optimally.
9. Alice plays the violin and piano, and would like to create a practice schedule. She will only practice one instrument on a given day, she can have break days when she does not practice any instrument, and she wants to make sure she does not neglect any instrument for more than two days (e.g. if she does not practice piano for two days, she must practice piano the next day). Following these rules, how many ways can she schedule her practice for eight days?

10. The chromatic musical scale repeats in groups of 12 pitches: C , $C\sharp$, D , $D\sharp$, E , F , $F\sharp$, G , $G\sharp$, A , $A\sharp$, and B (after B , the next note is C). Define a *chord* as a set of two or more distinct pitches. A *transposition* is a translation of the chord's pitches. Two chords are considered equivalent if they can be obtained from one another through a transposition. For instance, the chords $\{C, E, G, A\sharp\}$ and $\{C\sharp, D\sharp, G, A\sharp\}$ are equivalent because the second chord can be obtained from the first through a transposition of three steps ($C \rightarrow D\sharp$, $E \rightarrow G$, $G \rightarrow A\sharp$, $A\sharp \rightarrow C\sharp$, and the order of the notes does not matter). Find the number of distinct chords that can be formed from the set of all twelve pitches.