1. In $\triangle ABC$, the altitude to $\overline{AB}$ from C partitions $\triangle ABC$ into two smaller triangles, each of which is similar to $\triangle ABC$. If the side lengths of $\triangle ABC$ and of both smaller triangles are all integers, find the smallest possible value of AB.

   Answer: 25

   Solution: Let the altitude from C to AB intersect $\overline{AB}$ at D. Note that $\angle A$ and $\angle B$ must be acute since AD partitions ABC into two triangles. Since triangles ADC and ADB each contain right angles, we conclude that $\angle ACB$ must be right.

   Now, we must have $\triangle ABC \sim \triangle ACD \sim \triangle CBD$. We now have enough information to determine AD, AB, and CD in terms of AB, BC, and CA, which we denote as c, a, and b, respectively. We have

$$\frac{b}{AD} = \frac{c}{b} \implies AD = \frac{b^2}{c},$$
$$\frac{a}{BD} = \frac{c}{a} \implies BD = \frac{a^2}{c},$$
$$\frac{1}{2}CD \cdot c = \frac{1}{2}ab \implies CD = \frac{ab}{c}.$$

   Obviously, (a, b, c) has to be in the form (kx, ky, kz) where (x, y, z) is a Pythagorean triple with no common factors, and k is a positive integer. Note that in particular x, y, and z must be pairwise coprime, because of the constraint $x^2 + y^2 = z^2$.

   Given this, we need to find k such that

$$kz \mid k^2y^2 \iff z \mid ky^2 \iff z \mid k,$$

   so the smallest such k is when $k = z$. This choice makes BD and CD integral as well, so given any Pythagorean triple (x, y, z) with pairwise coprime entries, the minimum k required equals z, and the minimum possible value of c is z^2. The smallest such Pythagorean triple is $(3, 4, 5)$, so report $\boxed{25}$.

2. Four points O, A, B, and C satisfy $OA = OB = OC = 1$, $\angle AOB = 60^\circ$, and $\angle BOC = 90^\circ$. B is between A and C (i.e. $\angle AOC$ is obtuse). Draw three circles O_a, O_b, and O_c with diameters OA, OB, and OC, respectively. Find the area of region inside O_b but outside O_a and O_c.

   Answer: $(1 + \sqrt{3})/8$

   Solution: Let D, E, and F be the midpoints of BC, CA, and AB, respectively. Observe that O_a goes through E and F, O_b goes through D and F, and O_c goes through D and E; the radii to these points are all midlines of some triangle (either AOB, AOC, or BOC) and are parallel to sides of length 1. Hence, the region $O_b \setminus (O_a \cup O_c)$ has four vertices D, E, F, and B: along DE and EF the boundary is concave with the shape of an arc of radius 1, while along FB and BD it is convex. But note that $DEFB$ is a parallelogram, so $DE = BF$. This implies that two arcs–one from D to E coming from O_c and the other from B to F coming from O_b–are congruent, so the convex region outside segment FB can be fit into the concave region inside segment DE. Thinking similarly for EF and DB we have that the area of $O_b \setminus (O_a \cup O_c)$ is precisely the area of parallelogram $DEFB$. Thus, the answer is $|ABC|/2 = \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{4} - \frac{1}{2}\sin 150^\circ\right) = \boxed{(1 + \sqrt{3})/8}$.

   This problem can also be solved by use of the Principle of Inclusion-Exclusion: in particular, the desired region has area of

$$O_b - (O_a \cap O_b + O_c \cap O_b - O_a \cap O_b \cap O_c),$$

where it is clear that $O_a \cap O_b \cap O_c = O_a \cap O_c$ and is thus easily computable.

3. Circles with centers O_1, O_2, and O_3 are externally tangent to each other and have radii 1, $\frac{1}{2}$, and $\frac{1}{4}$, respectively. Now for $i > 3$, let circle O_i be defined as the circle externally tangent to circles O_{i-1} and O_{i-2} with radius 2^{1-i} that is farther from O_{i-3}. As n approaches infinity, the area of triangle $O_1O_2O_n$ approaches the value A. Find A.

   Answer: $\mathbf{\frac{\sqrt{14}}{6}}$

   Solution: First, note that all triangles $O_iO_{i+1}O_{i+2}$ are similar (for $i \geq 1$). In particular, this implies that for all such i, $m\angle O_{i+1}O_{i+3}O_{i+2} \cong m\angle O_iO_{i+2}O_{i+1}$ and $m\angle O_{i+3}O_{i+1}O_{i+2} \cong m\angle O_{i+4}O_{i+2}O_{i+3}$. Hence, $\angle O_iO_{i+2}O_{i+4} = \pi$ i.e. the points are collinear.

   From here, there are many ways to proceed. One way is to note that the collinearity of all O_{2n} and all O_{2n+1} implies that the desired area is simply the infinite sum of areas $\sum_{i=1}^{\infty}(O_iO_{i+1}O_{i+2})$. $O_1O_2O_3$ has side lengths $\frac{3}{2}$, $\frac{5}{4}$, and $\frac{3}{4}$, so its area is $\frac{1}{16}$ the area of a 3-5-6 triangle, which by Heron's Formula is $\sqrt{7 \cdot 4 \cdot 2 \cdot 1} = 2\sqrt{14}$. Furthermore, for all $i > 1$, $O_iO_{i+1}O_{i+2}$ has $\frac{1}{4}$ the area of $O_{i-1}O_iO_{i+1}$, so the desired sum is geometric with first term $\frac{\sqrt{14}}{8}$ and common ratio $\frac{1}{4}$. Hence, report $\frac{\frac{\sqrt{14}}{8}}{1-\frac{1}{4}} = \boxed{\frac{\sqrt{14}}{6}}$.