MATH MINGLE: WEEK 2

1. Given a positive integer n, the number A_n consists of the following digits, reading from the left:

   n 1's, followed by one 0, followed by n 8's, followed by one 9.

   For example,

   $$A_2 = 110889 \qquad \text{and} \qquad A_6 = 11111108888889.$$

   Prove that all of the numbers A_n are perfect squares.
2. A warden is in charge of 100 prisoners, each of which possesses one white glove and one black glove. The warden decides to play the following game: on each prisoner's head he places a hat on which is written an arbitrary distinct real number. The prisoners may see the numbers written on all hats except their own. Then, simultaneously, each prisoner chooses to put one of their gloves on their left hand and one on their right. The warden then lines the prisoners up side-by-side with the numbers on their hats in increasing order, and the prisoners link hands (in the natural way — no arm contortions). If every pair of linked hands is wearing gloves of the same color, the prisoners go free; otherwise they are executed. The prisoners may strategize beforehand but no further communication occurs once everyone has seen the hats. What should the prisoners do?
3. a. You have an 8×8 chessboard, and each square is colored either black or white. In one move, you may take any 3×3 or 4×4 square in the board and invert all the colors within it. Given an arbitrary starting position, can you always make the board all white in a finite number of moves?

   b. What if you instead have an $n \times n$ chessboard?
4. Given any positive integer k, construct the following number:

   $$S = 0.[2k][3k][5k][7k]\ldots,$$

   where $[m]$ denotes the decimal expansion of m and, each time, the product is taken between the next number and k. (For example, when $k = 2$, $S = 0.461014\ldots$, by concatenating the sequence 2×2, 3×2, 5×2, 7×2, $\ldots$)

   For which k is S rational?
5. Let n be a positive integer. Determine all entire functions f that satisfy, for all complex s and t, the functional equation

   $$f(s+t) = \sum_{k=0}^{n} f^{(n-1-k)}(s) f^{(k)}(t).$$

   (Here, $f^{(m)}$ denotes the m^{th} derivative of f.)
6. Show that the equation

   $$\left\lfloor \frac{n+1}{\phi} \right\rfloor = n - \left\lfloor \frac{n}{\phi} \right\rfloor + \left\lfloor \frac{\lfloor n/\phi \rfloor}{\phi} \right\rfloor - \left\lfloor \frac{\left\lfloor \frac{\lfloor n/\phi \rfloor}{\phi} \right\rfloor}{\phi} \right\rfloor + \left\lfloor \frac{\left\lfloor \frac{\left\lfloor \frac{\lfloor n/\phi \rfloor}{\phi} \right\rfloor}{\phi} \right\rfloor}{\phi} \right\rfloor - \cdots$$

   holds for every nonnegative integer n if and only if $\phi = (1+\sqrt{5})/2$.