1. Compute
\[
\frac{5 + \sqrt{6}}{\sqrt{2} + \sqrt{3}} + \frac{7 + \sqrt{12}}{\sqrt{3} + \sqrt{4}} + \cdots + \frac{63 + \sqrt{992}}{\sqrt{31} + \sqrt{32}}
\]

**Answer:** \(126\sqrt{2}\)

**Solution:** Rationalizing the denominators turns the numerators into differences of cubes, which gives
\[
3\sqrt{3} - 2\sqrt{2} + 4\sqrt{4} - 3\sqrt{3} + \cdots + 32\sqrt{32} - 31\sqrt{31} = 32\sqrt{32} - 2\sqrt{2}
\]
\[
= 128\sqrt{2} - 2\sqrt{2}
\]
\[
= 126\sqrt{2}
\]

2. Find the sum of the solution(s) \(x\) to the equation
\[
x = \sqrt{2022 + \sqrt{2022 + x}}.
\]

**Answer:** \(\frac{1 + \sqrt{8089}}{2}\)

**Solution:** Consider the following equations:
\[
y = \sqrt{2022 + y}
\]
\[
x = \sqrt{2022 + \sqrt{2022 + x}}
\]
Equation (*) and Equation (**) have the same solution, since if you plug the definition into the RHS repeatedly, replacing \(y\) for the first equation \(\sqrt{2022 + y}\) and replacing \(x\) in the second equation with \(\sqrt{2022 + x}\), then you arrive at \(x = \sqrt{2022 + \sqrt{2022 + \sqrt{2022}}\ldots} = y\). To solve for (*), we simply square on both sides and solve the quadratic. We discard the extraneous solution to get \(\frac{1 + \sqrt{8089}}{2}\).

3. Compute \(\left| \frac{1}{2022} + \frac{1}{2023} + \cdots + \frac{1}{2064} \right|\).

**Answer:** 47

**Solution:** Note that
\[
\frac{1}{2022} > \frac{1}{2023} > \frac{1}{2024} > \cdots > \frac{1}{2064}.
\]
Therefore,
\[
\frac{1}{2022} + \frac{1}{2023} + \cdots + \frac{1}{2064} < 43 \cdot \frac{1}{2022},
\]
which implies that
\[
\frac{1}{\frac{1}{2022} + \frac{1}{2023} + \cdots + \frac{1}{2064}} > \frac{1}{43 \cdot \frac{1}{2022}} = 2022 \cdot \frac{1}{43} = 47 \frac{1}{43}.
\]
Similarly,
\[
\frac{1}{2064} < \frac{1}{2022} + \frac{1}{2023} + \cdots + \frac{1}{2064}.
\]
Therefore,
\[
\frac{1}{2022} + \frac{1}{2023} + \cdots + \frac{1}{2064} > 43 \cdot \frac{1}{2064}
\]
which implies that
\[
\frac{1}{2022} + \frac{1}{2023} + \cdots + \frac{1}{2064} < \frac{1}{43} \cdot \frac{1}{2064} = \frac{2064}{43} = 48.
\]
Therefore, since
\[
\frac{1}{2022} + \frac{1}{2023} + \cdots + \frac{1}{2064} < 48,
\]
we have
\[
\left\lfloor \frac{1}{2022} + \frac{1}{2023} + \cdots + \frac{1}{2064} \right\rfloor = 47
\]

4. Let the roots of
\[
x^{2022} - 7x^{2021} + 8x^2 + 4x + 2
\]
be \(r_1, r_2, \cdots, r_{2022}\), the roots of
\[
x^{2022} - 8x^{2021} + 27x^2 + 9x + 3
\]
be \(s_1, s_2, \cdots, s_{2022}\), and the roots of
\[
x^{2022} - 9x^{2021} + 64x^2 + 16x + 4
\]
be \(t_1, t_2, \cdots, t_{2022}\). Compute the value of
\[
\sum_{1 \leq i, j \leq 2022} r_is_j + \sum_{1 \leq i, j \leq 2022} s_it_j + \sum_{1 \leq i, j \leq 2022} t_ir_j.
\]

**Answer:** 191

**Solution:** We wish to compute
\[
\sum_{1 \leq i, j \leq 2022} r_is_j + \sum_{1 \leq i, j \leq 2022} s_it_j + \sum_{1 \leq i, j \leq 2022} t_ir_j
\]
\[
= \frac{1}{2}((r_1 + r_2 + \cdots + r_{2022} + s_1 + s_2 + \cdots + s_{2022} + t_1 + t_2 + \cdots + t_{2022})^2
\]
\[
- (r_1^2 + r_2^2 + \cdots + r_{2022}^2 + s_1^2 + s_2^2 + \cdots + s_{2022}^2 + t_1^2 + t_2^2 + \cdots + t_{2022}^2))
\]
\[
- \sum_{1 \leq i < j \leq 2022} r_ir_j - \sum_{1 \leq i < j \leq 2022} s_is_j - \sum_{1 \leq i < j \leq 2022} t_it_j.
\]
We have \(r_1^2 + r_2^2 + \cdots + r_{2022}^2 = (r_1 + r_2 + \cdots + r_{2022})^2 - 2\sum_{1 \leq i < j \leq 2022} r_ir_j\). Substituting this in for each of \(r, s,\) and \(t\) gives us
\[
= \frac{1}{2}((r_1 + r_2 + \cdots + r_{2022} + s_1 + s_2 + \cdots + s_{2022} + t_1 + t_2 + \cdots + t_{2022})^2
\]
\[
- (r_1^2 + r_2^2 + \cdots + r_{2022}^2) - (s_1^2 + s_2^2 + \cdots + s_{2022}^2) - (t_1^2 + t_2^2 + \cdots + t_{2022}^2).
\]
Using Vieta’s formulas, this is equal to
\[
\frac{1}{2}((7 + 8 + 9)^2 - 7^2 - 8^2 - 9^2) = 7 \cdot 8 + 8 \cdot 9 + 9 \cdot 7 = 191.
\]
5. $x$, $y$, and $z$ are real numbers such that $xyz = 10$. What is the maximum possible value of $x^3y^3z^3 - 3x^4 - 12y^2 - 12z^4$?

**Answer:** 760

**Solution:** We can use the AM-GM inequality to minimize $3x^4 + 12y^2 + 12z^4$, which will maximize the overall expression. To make all the exponents the same on the geometric mean side, we split $12y^2$ into $6y^2 + 6y^2$. We have

$$3x^4 + 6y^2 + 6y^2 + 12z^4 \geq 4\sqrt[4]{1296x^4y^4z^4} = 24xyz = 240.$$ 

So,

$$x^3y^3z^3 - 3x^4 - 12y^2 - 12z^4 \leq 1000 - 240 = 760.$$ 

6. Compute $\cot \left( \sum_{n=1}^{23} \cot^{-1} \left( 1 + \sum_{k=1}^{n} 2k \right) \right)$.

**Answer:** $\frac{25}{23}$

**Solution:** Let the sum inside the cot be $S$. Then, we have

$$S = \sum_{n=1}^{23} \cot^{-1} (1 + n(n + 1)).$$

Note that $\cot^{-1} a - \cot^{-1} b = \tan^{-1} a - \tan^{-1} b = \tan^{-1} \left( \frac{\frac{1}{a} - \frac{1}{b}}{1 + \frac{1}{ab}} \right) = \tan^{-1} \left( \frac{b-a}{ab+1} \right) = \cot^{-1} \left( \frac{ab+1}{b-a} \right)$, so $\cot^{-1} (1 + n(n + 1)) = \cot^{-1} \frac{1}{n} - \cot^{-1} \frac{1}{n+1}$. Telescoping, our sum becomes

$$S = \cot^{-1} \frac{1}{1} - \cot^{-1} \frac{1}{24} = \cot^{-1} \frac{25}{23},$$

which gives $\cot S = \frac{25}{23}$.

7. Let $M = \{0, 1, 2, ..., 2022\}$ and let $f : M \times M \to M$ such that for any $a, b \in M$,

$$f(a, f(b, a)) = b$$

and $f(x, x) \neq x$ for each $x \in M$. How many possible functions $f$ are there (mod 1000)?

**Answer:** 0

**Solution:** No such functions $f$ exist.

Suppose otherwise. Write $f(b, a) = c$. Then by the condition in the problem, $f(a, c) = b$ and $f(c, b) = f(c, f(a, c)) = a$. Consider the set

$$S = \{(x, y, z) \mid f(x, y) = z, \ x, y, z \in M\}$$

By our observation above, $(x, y, z) \in S$ if and only if $(y, z, x) \in S$. Hence we may partition $S$ into set of the form $\{(x, y, z), (y, z, x), (z, x, y)\}$ for $x, y, z$ not all equal (since it is known $(x, x, x) \notin S$). Hence 3 divides $|S|$. Then, $|S| = |M|^2 = 2023^2$, as to form elements in $S$, we can arbitrarily choose $x$ and $y$ while $z$ is then determined. This is a contradiction as 3 does not divide $2023^2$. 

8. For all positive integers $m > 10^{2022}$, determine the maximum number of real solutions $x > 0$ of the equation $m x = \left\lfloor x^{11/10} \right\rfloor$.

**Answer:** 10

**Solution:** We claim that there can never be more than 10 solutions. Clearly, $x$ will never be very small, since $m$ is so large. Let $x = (k^{10} + \epsilon)$ for some small $\epsilon$ and some positive integer $k$ as large as possible. Then we may Taylor Expand

$$k^{11} < m(k^{10} + \epsilon) < (k^{10} + \epsilon)^{11/10} = k^{11} + \frac{11}{10} k\epsilon + \frac{1}{2} \cdot \frac{11}{10} \cdot \frac{1}{10} k^{-9} \epsilon^2 + \cdots$$

It is clear that $\lfloor x^{11/10} \rfloor = \lfloor k^{11} + \frac{11}{10} k\epsilon \rfloor$, since because $k$ is much larger than $\epsilon$, the remaining terms in the summation will be negligible decimals. Using the fact that $k$ is an integer, we may write

$$m(k^{10} + \epsilon) = \left( k^{11} + \frac{11}{10} k\epsilon \right) = k^{11} + \frac{11}{10} k\epsilon - \left\{ \frac{11}{10} k\epsilon \right\}$$

Now since the LHS is on the order of $k^{10}$, it must in fact be 0 (we could obtain this by bounding it below my removing the epsilon and getting a contradicting inequality), and so

$$\frac{1}{10} k\epsilon = \left\{ \frac{11}{10} k\epsilon \right\}$$

Therefore, $\frac{11}{10} k\epsilon - \left\lfloor \frac{11}{10} k\epsilon \right\rfloor = k\epsilon$ and so $\left\lfloor \frac{11}{10} k\epsilon \right\rfloor = k\epsilon$, and so $k\epsilon$ is an integer. Taking the floor of the expression above, we find that $\left\lfloor \frac{11}{10} k\epsilon \right\rfloor = 0$. Therefore, $k\epsilon$ can only take on the values 0, 1, 2, \ldots, 9, and we have achieved 10 solutions, as desired.

We can additionally show this is achievable. It is not hard to see that $x = (10^{2022} + 1)^{10} + \frac{r}{10^{2022} + 1}$ for $r = 0, 1, 2, \ldots, 9$ all satisfy this equation for $m = 10^{2022} + 1$.

9. Let $P(x) = 8x^3 + ax^2 + bx + 1$ for $a, b \in Z$. It is known that $P$ has a root $x_0 = p + \sqrt{q} + \sqrt[3]{r}$, where $p, q, r \in Q, q \geq 0$; however, $P$ has no rational roots. Find the smallest possible value of $a + b$.

**Answer:** –6

**Solution:** We have $P(x_0) = 0$ and $x_0 = p + \sqrt{q} + \sqrt[3]{r}$. Note that $P \in Q[x]$ (since $Q[x] \equiv Z[x]$) and $\deg P = 3$. Moreover, observe that if $r = 0$, $P$ has at least one rational root, hence $r \neq 0$. Now consider the polynomial

$$Q(x) = (x - p - \sqrt{q})^3 - r.$$  

Trivially, $Q(x_0) = 0$, and $Q \in Q[\sqrt[3]{q}]$, i.e. the coefficients of $Q$ are of the form $\alpha + \beta \sqrt{q}$ for $\alpha, \beta \in Q$. Since $\deg Q = 3$, we can express $P$ in terms of $Q$ as

$$P(x) = 8Q(x) + R(x)$$

where $R \in Q[\sqrt[3]{q}], \deg R \leq 2$. Since $P(x_0) = Q(x_0)$, it follows that $R(x_0) = 0$. We consider cases for the degree of $R(x)$:
• Case 1: $\deg R = 2$. If $R \mid P$, then $\frac{P}{R} \in \mathbb{Q}[q]$ and is of degree 1, hence $P$ has a rational zero, which is a contradiction. If $R \nmid P$, we can divide with remainder to get

$$P(x) = c(x - x_r)R(x) + S(x).$$

We now have that $S \in \mathbb{Q}[^\sqrt{q}]$ and $S(x_0) = 0$, hence $x_0 \in \mathbb{Q}[^\sqrt{q}]$ (deg $S = 1$), which is again a contradict.

• Case 2: $\deg R = 1$. Since $R(x_0) = 0$, as above, $x_0 \in \mathbb{Q}[^\sqrt{q}]$ (deg $S = 1$), which is a contradiction.

We therefore have that $\deg R = 0$, and we can consider without loss of generality that $P(x) = 8Q(x)$ (if $R(x) = c$, $c$ can be absorbed by $r$). Since the coefficients of $P$ are integers, $q = 0$. Expanding $Q$ we therefore get

$$P(x) = 8Q(x) = 8x^3 - 3 \cdot 8x^2p + 3 \cdot 8xp^2 - 8p^3 - 8r$$

where all coefficients are integers and $8p^3 + 8r = -1$. We seek the minimum value of $a + b = 24(\alpha + 1)(\alpha)$. The minimum of the function is at $\alpha = \frac{1}{2}$, for which we find $r = -\frac{1}{4}$ and $P(x) = 8x^3 - 12x^2 + 6x + 1 = (2x - 1)^3 + 2$. Finally, $a + b = -6$.

10. Let $f^1(x) = x^3 - 3x$. Let $f^n(x) = f(f^{n-1}(x))$. Let $\mathcal{R}$ be the set of roots of $\frac{f^{2022}(x)}{x}$. If

$$\sum_{r \in \mathcal{R}} \frac{1}{r^2} = \frac{a^b - c}{d}$$

for positive integers $a, b, c, d$, where $b$ is as large as possible and $c$ and $d$ are relatively prime, find $a + b + c + d$.

**Answer:** 4072

**Solution:** Consider the substitution $x = 2 \cos t$. From this we obtain that $f^1(2 \cos t) = 8 \cos^3 t - 6 \cos t = 2(4 \cos^3 t - 3 \cos t) = 2 \cos 3t$. Thus $f^{2022}(2 \cos t) = 2 \cos 3^{2022}t$. Therefore $f^{2022}(x)$ is the $3^{2022}$th Dickson Polynomial and has the form

$$f^{2022}(x) = \sum_{k=0}^{3^{2022} - 1} \frac{3^{2022} - k}{3^{2022} - k} x^{3^{2022} - 2k}$$

The roots of the polynomial $f^{2022}(x)/x$ are all of those of $f^{2022}(x)$ except 0 (which we are guaranteed to have from parity).

We are left to determine the coefficient of $x$ and $x^3$ in $f^{2022}(x)$ and then we can finish by Vieta’s. For simplicity, let $3^{2022} = 2\alpha + 1$.

The linear coefficient is

$$c_1 = \frac{2\alpha + 1}{\alpha + 1} \frac{\alpha + 1}{\alpha} (-1)^\alpha = (2\alpha + 1)(-1)^\alpha$$

The coefficient of $x^3$ is

$$c_3 = \frac{2\alpha + 1}{\alpha + 2} \frac{\alpha + 2}{\alpha - 1} (-1)^\alpha - 1 = \frac{(2\alpha + 1)(\alpha + 1)(\alpha)(-1)^{\alpha - 1}}{6}$$
By Vieta’s, we wish to find 
$$-\frac{2c_3}{c_1}$$
so our desired answer is \(\frac{(\alpha+1)(\alpha)}{6}\), and because \(\alpha = \frac{3^{2022} - 1}{2}\) we have

\[
\left(\frac{3^{2022} - 1}{2}\right) \left(\frac{3^{2022} + 1}{2}\right) = \frac{3^{4044} - 1}{24}
\]

which gives us an answer of \(3 + 4044 + 1 + 24 = 4072\).