1. Let \( a \) be the base 10 integer equivalent to \( 2021_{20} \) and let \( b \) be the base 10 integer equivalent to \( 2021_{21} \). Compute \( b - a \).

Answer: 2524

Solution: Converting to base 10,
\[
b - a = (2 \cdot 21^3 + 2 \cdot 21 + 1) - (2 \cdot 20^3 + 2 \cdot 20 + 1) \\
= 2(21^3 - 20^3) + 2(21 - 20) \\
= 2(1261) + 2(1) \\
= 2524.
\]

2. Let \( f \) be a function such that for any positive integer \( n \), \( f(n) \) is equal to the median of the positive factors of \( n \). Compute the sum of all positive integers \( n \) such that \( 20 < f(n) < 21 \).

Answer: 2856

Solution: Bash pairs that sum to 41 to get the solution set \( \{148, 310, 348, 390, 414, 418, 420\} \) which sums to 2856.

3. Let \( k \) be a randomly chosen positive divisor of \( 20! \). What is the probability that \( k \) can be written as \( a^2 + b^2 \) for some integers \( a \) and \( b \)?

Answer: \( \frac{5}{54} \)

Solution: The choice \( k \) can be written as the sum of two squares if and only if every prime \( p \equiv 3 \pmod{4} \) appears to an even power in the prime factorization of \( k \). The primes \( p \leq 20 \) with \( p \equiv 3 \pmod{4} \) are 3, 7, 11, and 19. Also, \( 20! = 3^8 \cdot 7^2 \cdot 11^1 \cdot 19^1 \cdot k \) where \( k \) is not divisible by any of these primes. Therefore, a randomly positive divisor of \( 20! \) has prime factorization \( 3^a \cdot 7^b \cdot 11^c \cdot 19^d \cdot l \) where \( l \) is not divisible by any of these primes, \( a \in [0, 8], b \in [0, 2], c \in [0, 1], d \in [0, 1] \). Furthermore \( a, b, c, \) and \( d \) take on the possible values with equally likelihood and independently of each other. It suffices to compute the probability that \( a, b, c, \) and \( d \) are all even which is \( \left( \frac{5}{8} \right) \left( \frac{3}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{5}{54} \).