1. Find all integer solutions to 
\[
\frac{1}{\log_8 n} + \frac{1}{\log_n \frac{1}{4}} = \frac{-5}{2}
\]

Answer: 64

Solution: Applying \(\log_a b = \frac{\log_c b}{\log_c a}\) with \(c = 2\) on both fractions, this becomes:
\[
\frac{\log_2 8}{\log_2 n} + \frac{\log_2 n}{\log_2 \frac{1}{4}} = \frac{-5}{2}
\]
Substituting \(x = \log_2 n\) and evaluating \(\log_2 8 = 3\) and \(\log_2 \frac{1}{4} = -2\), this becomes:
\[
\frac{3}{x} + \frac{x}{-2} = \frac{-5}{2}
\]
Multiplying both sides by \(-2x\) yields \(-6 + x^2 = 5x\), which factors into \((x - 6)(x + 1) = 0\). This quadratic has solutions \(x = 6\) and \(x = -1\). When \(x = -1\), we have \(\log_2 n = -1 \implies n = \frac{1}{2}\), which is not an integer. When \(x = 6\), we have \(\log_2 n = 6 \implies n = 64\).

2. Let \(f : \mathbb{N} \to \mathbb{N}\) be a bijection such that for all \(a\), either \(a = f(a)^2\) or \(f(a) = a^2\). For how many numbers less than 40 is \(f(a) \neq a^2\)?

Answer: 4

Solution: We start by figuring out the pattern. We can see that \(f(0) = 0\) and \(f(1) = 1\). Then \(f(2) = 4\) and \(f(3) = 4\). Then we can see that \(f(4) = 2\) or else no other number satisfies \(f(a) = 2\) and this is a bijection. Similarly \(f(9) = 3\). Then \(f(16) = 16^2\) since \(f(2) = 4\) and we have a bijection. So, we can see that the only numbers where \(f(a) \neq a^2\) are 4, 9, 25, 36. So, there are \(4\) numbers.

3. Let \(x\) be a real number satisfying the equation \(x^2 - 3x + 1 = 0\). Then \(x^{16} - kx^8 + 1 = 0\) for some constant \(k\). Compute \(k\).

Answer: 2207.

Solution: Rearrange the given quadratic equation to \(x + \frac{1}{x} = 3\). Letting \(P_n(x) = x^n + \frac{1}{x^n}\), it is easy to derive the recursive formula
\[
P_{2n}(x) = P_n(x)^2 - 2.
\]
We know that \(P_1(x) = 3\), so we can apply the recursion to obtain:
\[
P_2(x) = P_1(x)^2 - 2 = 7,
P_4(x) = P_2(x)^2 - 2 = 47,
P_8(x) = P_4(x)^2 - 2 = 2207.
\]
Hence, \(x^8 + \frac{1}{x^8} = 2207\), and rearranging yields \(x^{16} - 2207x^8 + 1 = 0\). Thus, \(k = 2207\).