1. Pentagon $ABCDE$ has $AB = BC = CD = DE$, $\angle ABC = \angle BCD = 108^\circ$, and $\angle CDE = 168^\circ$. Find the measure of angle $\angle BEA$ in degrees.

**Answer:** 24

**Solution:** Construct point $F$ such that $\angle CDF = 108^\circ$, $\angle FDE = 60^\circ$, and $DF = DE$. Then triangle $FED$ is equilateral. Note that $BE$ is a line of symmetry for hexagon $ABCDEF$, so $\angle BEF = 30^\circ$. Furthermore, since triangle $AFE$ is isosceles we see that $\angle AEF = 6^\circ$. Thus $\angle BEA = \angle BEF - \angle AEF = 24^\circ$.

2. On each edge of a regular tetrahedron, five points that separate the edge into six equal segments are marked. There are twenty planes that are parallel to a face of the tetrahedron and pass through exactly three of the marked points. When the tetrahedron is cut along each of these twenty planes, how many new tetrahedrons are produced?

**Answer:** 76

**Solution:** Consider the five cuts in the same direction, which cut the tetrahedron into five frustums of different size and a unit regular tetrahedron. Note that the $i$-th smallest frustum has a smaller base being an equilateral triangle with side length $i$ and a larger base being an equilateral triangle with side length $i + 1$. Now perform the cuts in the remaining directions on each frustum. Below is an example for the cuts on second smallest frustum.

After that, on the smaller base there are $\frac{i(i-1)}{2}$ downward pointing triangles and each of them faces an intersection on the larger base, meaning that each correspond to a tetrahedron; on the larger base there are $\frac{(i+1)(i+2)}{2}$ upward-pointing triangles, each facing an intersection and therefore corresponding to a tetrahedron as well. The $\frac{i(i+1)}{2}$ upward-pointing triangles on the smaller base and downward-pointing triangles on the larger base correspond to each other and each pair correspond to a regular octahedron. For example, when we separate all the pieces in the second frustum, we should get the following pieces: (in their relative position in order to aid visualization)
Finally, by summing $\frac{i(i-1)}{2}$ and $\frac{(i+1)(i+2)}{2}$ over $i = 0, 1, 2, 3, 4, 5$, we get that there are $(0 + 0 + 1 + 3 + 6 + 10) + (1 + 3 + 6 + 10 + 15 + 21) = 76$ tetrahedrons (and $0 + 1 + 3 + 6 + 10 + 15 = 35$ octahedrons).

3. Three cities that are located on the vertices of an equilateral triangle with side length 100 units. A missile flies in a straight line in the same plane as the equilateral triangle formed by the three cities. The radar from City $A$ reported that the closest approach of the missile was 20 units. The radar from City $B$ reported that the closest approach of the missile was 60 units. However, the radar for city $C$ malfunctioned and did not report a distance. Find the minimum possible distance for the closest approach of the missile to city $C$.

**Answer:** $30\sqrt{3} - 20$

**Solution:** We first begin by drawing a circle of radius 20 around city $A$ and a circle of radius 60 around city $B$. The path of the missile must be one of the four (interior or exterior) tangents to these two circles. We will focus on calculating the distance from $C$ to the nearest tangent, since the distance from $C$ to the other tangents can be calculated similarly.

Let the tangent intersect segment $AB$ at $D$, and let the tangent point of the circle centered at $A$ touch the line at $A'$ and the tangent point of the circle centered at $B$ touch the line at $B'$. Then, triangle $BB'D$ is similar to triangle $AA'D$ with ratio $1:3$. Therefore, $AD = 25$ and $DB = 75$ and we see that the triangles are in fact $3-4-5$ right triangles.

Now, let the path of the missile intersect segment $BC$ at $E$, and draw a circle around $C$ such that the circle is also tangent to the path of the missile. Once again, denote the point of tangency as $C'$. Then, triangle $CC'E$ is similar to $BB'E$. Let $CC' = r$ be the radius of the circle around $C$ (which is also the answer we want).

We can then write the equation $\frac{r}{\cos(60^\circ - \theta)} + \frac{60}{\cos(60^\circ - \theta)} = 100$ where $\theta = \angle B'B'D$. By using angle sum formulas, this reduces down to $r + 60 = 100\cos(60^\circ - \theta) = 40 + 30\sqrt{3}$. Our desired distance is thus $r = 30\sqrt{3} - 20$. 