1. Suppose \( 2 + \frac{1}{5 + \frac{1}{2 + \frac{1}{5 + \ldots}}} = \frac{a+\sqrt{5}}{c} \), where \( a, b, \) and \( c \) are positive integers, and \( b \) is not divisible by the square of any prime. Compute \( a + b + c \).

**Answer:** 45

**Solution:** If \( x \) is the requested continued fraction, note \( x = 2 + \frac{1}{5 + \frac{1}{x}} \), so \( x = \frac{11x + 2}{5x + 1} \) \( \Rightarrow 5x^2 - 10x - 2 = 0 \). Taking the positive root, we find \( x = \frac{5 + \sqrt{55}}{5} \), in which case \( a + b + c = 5 + 35 + 5 = \boxed{45} \).

2. Let \( a, b, c \) be the roots of the polynomial \( 4x^3 + 24x^2 - 237x + 2 \). Find the value of \( a^2(a+1) + b^2(b+1) + c^2(c+1) \).

Write your answer as a decimal rounded to the nearest tenth.

**Answer:** \(-1129.5\)

**Solution:** Notice that \( a^2(a+1)+b^2(b+1)+c^2(c+1) = (a^3+b^3+c^3)+(a^2+b^2+c^2) \). Using Vieta’s formulas we know that \( a+b+c = -24/4 = -6 \), \( ab+bc+ca = -59.25 \), and \( abc = -24/4 = -0.5 \). Then,

\[
a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = (-6)^2 - 2(-\frac{237}{4}) = 154.5
\]

Also,

\[
a^3 + b^3 + c^3 = (a^2 + b^2 + c^2)(a + b + c) - (ab + bc + ca)(a + b + c) + 3abc = 154.5(-6) - (-59.25)(-6) + 3(-0.5) = -1284
\]

Therefore,

\[
a^2(a+1) + b^2(b+1) + c^2(c+1) = 154.5 - 1284 = \boxed{-1129.5}
\]

3. Suppose \( S \) is a set of functions with the property that, if \( f(x) \) and \( g(x) \) are in \( S \), then \( (f \circ g)(x) = f(g(x)) \) is in \( S \). Given that the functions \( r(x) = \frac{\sqrt{3} + 1}{\sqrt{3} - x} \) and \( s(x) = \frac{1}{x} \) are in \( S \), compute the smallest possible size of \( S \).

**Answer:** 12

**Solution:** Let the notation \( f^n(x) \) denote repeated composition of the same function, so \( f^1(x) = f(x) \), \( f^2(x) = f(f(x)) \), and so on. Note \( r^6(x) = x \) and \( s^2(x) = x \) (in particular, 6 and 2 are the smallest integers such that \( r^n(x) = x, s^n(x) = x \)). In addition, \( (s \circ r \circ s \circ r)(x) = x \). Then we can compute compositions of functions that avoid reduction of one or more terms in the composition into the identity function. This determines that \( S \) must have a minimum size of 12. The functions are \( id, r, r^2, r^3, r^4, r^5, s, s \circ r, s \circ r^2, s \circ r^3, s \circ r^4, s \circ r^5 \).

Alternatively, we can consider the functions geometrically as symmetries of a regular hexagon with labeled vertices. If \( r \) represents clockwise rotation of 60° about the center and \( s \) represents reflection of the hexagon about a fixed line of symmetry (so that applying \( r \) six times, \( s \) twice,
or $r$, then $s$, then $r$, then $s$, gives us back the vertices in their original orientation), it is easy to see that there are 12 distinct orientations of the labeled hexagon, which correspond to the minimum $\lfloor 12 \rfloor$ functions in $S$. 