Comment: Version 1.1

RH12

1. Evaluate $\sqrt{2223^2 - 8888}$.

Answer: 2221

Solution: Note that $\sqrt{2223^2 - 8888} = \sqrt{(2222 + 1)^2 - 8888} = \sqrt{2222^2 + 4444 + 1 - 8888} = \sqrt{2222^2 - 4444 + 1} = \sqrt{(2222 - 1)^2} = 2221$.

RH11

2. If $a$ is the only real number that satisfies $\log_{2020} a = 2020^{2020-a}$ and $b$ is the only real number that satisfies $2020^b = 2020^{2020-b}$, what is the value of $a+b$?

Answer: 202020

Solution: We have $2020^{2020-a} = a$ and $2020^b = 2020^{2020-b}$. If $x = 2020^{2020-a}$, then $2020^x = 2020^{2020-x}$. Since $b$ is the only real number that satisfies $2020^b = 2020^{2020-b}$, $x$ must be equal to $b$. Therefore, $a+b = 202020$.

SS04

3. Two cars driving from city A to city B leave at the same time. The first car drives at some constant speed during the whole trip. The second car travels at a speed 12 km/hr slower than the first car until the halfway point between city A and city B. After the halfway point, the second car travels at a constant speed of 72 km/hr. Both cars end up reaching city B at the same time. Calculate the speed of the first car in km/hr, given that it was faster than 40 km/hr.

Answer: 48

Solution: Let $d$ be the distance between the cities and $s$ be the speed of the first car in km/hrs. Then $\frac{d}{s}$ is the total time taken by the first car to get from city A to B. For the second car, $\frac{d/2}{s-12}$ is the time taken to get from city A to the halfway point and $\frac{d/2}{72}$ is the time taken to get from the halfway point to city B. Therefore,

$$\frac{d}{s} = \frac{d/2}{s-12} + \frac{d/2}{72}.$$

Dividing by $d/2$ and multiplying by the denominators,

$$144(s - 12) = 72s + s(s - 12).$$

Rearranging,

$$s^2 - 84s + 1728 = (s - 36)(s - 48) = 0,$$

so $s$ must be either 36 or 48. However, given that the speed was greater than 40 km/hr, $s$, the speed of the first car in km/hrs, must be 48.

KW47

4. Find the value of $bc$ such that $x^2 - x + 1$ divides $20x^{11} + bx^{10} + cx^9 + 4$.

Answer: $-480$

Solution: First we consider a polynomial $p(x) = p_0 x^9 + p_9 x^8 + \ldots + p_0$ such that $p(x)(x^2 - x + 1) = ax^{11} + bx^{10} + cx^9 + 4$. Clearly $p_0 = 4$. Then we can deduce since $-4x + p_1 x = 0$ that $p_1 = 4$. Then we continue to see that $p_2 x^2 - 4x^2 + 4x^2 = 0$, so $p_2 = 0$. Continuing this pattern we have that $p_3 = -4$, $p_4 = -4$, $p_5 = 0$, $p_6 = 4$, $p_7 = 4$, and $p_8 = 0$. Then $p_9 x^9 + 4p_7 x^9 = cx^9$, so $p_9 = c - 4$. However, we can see that since $x^2 - x + 1$ is monic, $a = p_9$. So, $a = c - 4$. In addition, $b = -a$. So, $bc = -480$. 
5. Suppose $f(x)$ is a monic quadratic polynomial such that there exists an increasing arithmetic sequence $x_1 < x_2 < x_3 < x_4$ where $|f(x_1)| = |f(x_2)| = |f(x_3)| = |f(x_4)| = 2020$. Compute the absolute difference of the two roots of $f(x)$.

Answer: $10\sqrt{101}$

Solution: Suppose $f(x) = (x - h)^2 - k$ with vertex $(h, k)$ where $k$ is non-negative. Since the roots of $f(x)$ are $h \pm \sqrt{k}$, the absolute difference of the two roots of $f(x)$ is $2\sqrt{k}$. By symmetry, if $x_2 = h - d$, then $x_3 = h + d$, which means $x_1 = h - 3d$ and $x_4 = h + 3d$. Since $x_1 < x_2 < h$ and $f(x)$ is monic, $f(x_1) = f(x_4)$ must be positive and $f(x_2) = f(x_3)$ must be negative. Therefore, $f(x_1) = -f(x_2) = 2020$. Substituting, $9d^2 - k = -(d^2 - k)$, which implies $k = 5d^2$. Since $f(x_1) = 9d^2 - k = 4d^2 = 2020$, it follows that $k = 5d^2 = \frac{5}{4}(4d^2) = \frac{5}{4}(2020)$. Therefore, the absolute difference of the roots of $f(x)$ is $2\sqrt{k} = 5\sqrt{2020} = 10\sqrt{101}$.

6. Let $f: A \rightarrow B$ be a function from $A = \{0, 1, \ldots, 8\}$ to $B = \{0, 1, \ldots, 11\}$ such that the following properties hold:

$$f(x + y \mod 9) \equiv f(x) + f(y) \mod 12$$

$$f(xy \mod 9) \equiv f(x)f(y) \mod 12$$

for all $x, y \in A$. Compute the number of functions $f$ that satisfy these conditions.

Answer: 2

Solution: Note $f(n) \equiv f(1 + 1 + \ldots + 1) = n \times f(1) \mod 12$, so $f$ can be completely determined if we know $f(1)$. In addition, $f(0) = f(0 + 0) \equiv f(0) + f(0) \mod 12 \Rightarrow f(0) = 0$. Now consider $f(0) = 0 = f(1 + 1 + \ldots + 1) \equiv 9f(1) \mod 12$. Thus $f(1)$ equals 0, 4, or 8.

If $f(1) = 0$, then $f(n) = 0$ for all $n$, which is well-defined. If $f(1) = 4$, then $f(n) \equiv 4n \mod 12$, which is well-defined as $f(n) \equiv f(n + 9) = 4n + 36 \mod 12 \equiv 4n$, and $f(m) \times f(n) = 4m \times 4n \equiv 4mn \mod 12 = f(mn)$. However, if $f(1) = 8$, $f(1) = f(1) \times f(1) = 8 \times 8 \equiv 4 \mod 12$, so this $f$ doesn’t satisfy the second condition. So there are only $2$ functions that satisfy the given properties, which are the $f$ defined by $f(1) = 0$ and $f(1) = 4$.

7. Let $a_n$ be a sequence where $a_0 = \sqrt{3}$, $a_1 = \sqrt{2}$, $a_3 = -1$ (not $a_2$) and $a_n = a_{n-1}a_{n-2} - a_{n-3}$ for $n \geq 3$. Compute $a_{2020}$.

Answer: $\frac{\sqrt{6} + \sqrt{2}}{2}$

Solution: First, observe that $a_0 = 2\cos(30^\circ)$, $a_1 = 2\cos(45^\circ)$, $a_2 = \sqrt{6} - \sqrt{2} = 2\cos(75^\circ)$, $a_3 = 2\cos(120^\circ)$. Note that if $a_k = 2\cos(a - b)$, $a_{k+1} = 2\cos(b)$, $a_{k+2} = 2\cos(a)$, $a_{k+3} = 4\cos(a)\cos(b) - 2\cos(b - a) = 2\cos(b + a)$. Therefore, by a simple inductive argument, $a_n = 2\cos(15F_{n+3})$ where $F_n$ is the $n^{th}$ Fibonacci number. Since $\cos$ has a period of 360 degrees and the Fibonacci numbers mod 24 have a period of 24, it follows that the sequence has a period of 24. Therefore, it follows that $a_{2020} = a_4 = 2\cos(195^\circ) = -\frac{\sqrt{6} + \sqrt{2}}{2}$.

8. For how many integers $n$ with $3 \leq n \leq 2020$ does the inequality

$$\sum_{k=0}^{\lfloor(n-1)/4\rfloor} \binom{n}{4k+1}9^k > 3 \sum_{k=0}^{\lfloor(n-3)/4\rfloor} \binom{n}{4k+3}9^k$$

hold?

Answer: 672
Solution: By the binomial theorem, we have \((1 + \sqrt{3}i)^n = \sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k+1} (\sqrt{3})^{4k+1} - \sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k+3} (\sqrt{3})^{4k+3}\)
\[= \sqrt{3} \left( \sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k+1} 9^k - 3 \sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k+3} 9^k \right).\]

It follows that \(\sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k+1} 9^k > 3 \sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k+3} 9^k\) if and only if the imaginary part of \((1 + \sqrt{3}i)^n\) is positive. Since \(1 + \sqrt{3}i = 2(\cos(2\pi/3) + i \sin(2\pi/3))\), by de Moivre’s formula we have \((1 + \sqrt{3}i)^n = 2^n (\cos(2\pi n/3) + i \sin(2\pi n/3))\), and hence the imaginary part of \((1 + \sqrt{3}i)^n\) is positive if and only if \(\sin(2\pi n/3) > 0\), which is true if and only if \(n \equiv 1, 2 \mod 6\). So now we ask how many integers \(n\) between 3 and 2020, inclusive, satisfy \(n \equiv 1, 2 \mod 6\). Since 2016/6 = 336, we know there are \(2 \times 336 = 672\) integers between 1 and 2016, inclusive, with remainder 1 or 2. Thus there are 670 such integers between 3 and 2016, inclusive. Since 2017 and 2018 also have remainders 1 and 2, the total number of such integers between 3 and 2020 inclusive is \(672\).

WX07 9. A sequence of numbers is defined by \(a_0 = 2\) and for \(i > 0\), \(a_i\) is the smallest positive integer such that \(\sum_{j=0}^{i} \frac{1}{a_j} < 1\). Find the smallest integer \(N\) such that \(\sum_{i=0}^{N} \frac{1}{\log_2(a_i)} < \frac{1}{2^{2022}}\).

Answer: 2022

Solution: We claim that our sequence follows the recurrence \(a_{n+1} = a_n^2 - a_n + 1\).

First, we will show that \(\sum_{j=0}^{k} \frac{1}{a_j} = \frac{a_{k+1}^2 - a_k - 1}{a_{k+1}^2 - a_k}\) for all \(k\). This is obvious for \(k = 0\). In fact, we see that if this is true for all \(k\), then it implies our recurrence. Therefore, to finish, we just need to show that \(a_{k+1} = a_k^2 - a_k + 1\) implies that \(\sum_{j=0}^{k+1} \frac{1}{a_j} = \frac{a_{k+2}^2 - a_{k+1} - 1}{a_{k+2}^2 - a_{k+1}}\).

This is just a matter of brute force algebra.

\[
\sum_{j=0}^{k+1} \frac{1}{a_j} = \frac{a_{k+1}^2 - a_k - 1}{a_{k+1}^2 - a_k} + \frac{1}{a_{k+1}^2 - a_{k+1}} = \frac{a_{k+2}^2 - a_k - 1}{a_{k+2}^2 - a_{k+1}} + \frac{1}{a_{k+2}^2 - a_{k+1}}
\]

(To better understand what’s going on, let \(P(n)\) be the assertion that \(a_{n+1} = a_n^2 - a_n + 1\) and \(S(n)\) be the assertion that \(\sum_{j=0}^{n} \frac{1}{a_j} = \frac{a_{n+1}^2 - a_n - 1}{a_{n+1}^2 - a_n}\). It is easy to see that \(S(n) \implies P(n)\) and we have shown that \(P(n) \implies S(n+1)\). Since \(S(0)\) is easily seen to be true, we have a domino effect which shows that \(P(n)\) is true for all \(n\).)

Now, it is easy to show that by induction, \(2^{2^i - 1} \leq a_i \leq 2^{2^i}\) for all \(i \geq 0\). The motivation for this is noticing that the recurrence approximately grows as \(a_{n+1} \approx a_n^2\). Therefore, for any nonnegative integer \(k\),

\[
\frac{1}{2^{k-1}} = \sum_{i=k}^{\infty} \frac{1}{2^i} \leq \sum_{i=k}^{\infty} \frac{1}{\log_2(a_i)} \leq \sum_{i=k}^{\infty} \frac{1}{2^{i-1}} \leq \frac{1}{2^{k-2}}.
\]

Therefore, our answer is 2022.
10. Let \( f(a, b) \) be a third degree two-variable polynomial with integer coefficients such that \( f(a, a) = 0 \) for all integers \( a \) and the sum

\[
\sum_{a, b \in \mathbb{Z}^+ \atop a \neq b} \frac{1}{2f(a, b)}
\]

converges. Let \( g(a, b) \) be the polynomial such that \( f(a, b) = (a - b)g(a, b) \). If \( g(1, 1) = 5 \) and \( g(2, 2) = 7 \), find the maximum value of \( g(20, 20) \).

**Answer:** 43

**Solution:** Since we have that \( f(a, a) = 0 \) for all integers \( a \), we can write \( f(a, b) = (a - b) \cdot \left(m_0a^2 + m_1ab + m_2b^2 + m_3a + m_4b + m_5\right) \) for some integers \( m_0, m_1, m_2 \) since every term of \( f \) has degree 3. Thus, \( g(a, b) = m_0a^2 + m_1ab + m_2b^2m_3a + m_4b + m_5 \).

A necessary condition for the sum to converge is that \( g(n, n - 1) > 0 \) for all natural numbers \( n > N_1 \) for some \( N_1 \). Otherwise, \( f(n, n - 1) \) is negative for infinitely many pairs \( (n, n - 1) \) and the sum wouldn’t converge.

Similarly, \( g(n - 1, n) \) must be negative for all natural numbers \( n > N_2 \) for some \( N_2 \). This can be rewritten as

\[
m_0(n^2) + m_1(n)(n - 1) + m_2(n - 1)^2 + m_3(n) + m_4(n - 1) + m_5 > 0
\]

\[
m_0(n - 1)^2 + m_1(n - 1)(n) + m_2(n)^2 + m_3(n - 1) + m_4(n) + m_5 < 0.
\]

For all \( n > \max(N_1, N_2) \). In other words, we have:

\[
(m_0 + m_1 + m_2)n^2 - (m_1 + 2m_2 + m_3 + m_4)n + m_2 - m_4 + m_5 > 0
\]

\[
(m_0 + m_1 + m_2)n^2 - (m_1 + 2m_0 + m_3 + m_4)n + m_0 - m_3 + m_5 < 0.
\]

In order for this to be possible, we must have \( m_0 + m_1 + m_2 = 0 \). Therefore, \( g(a, a) = (m_3 + m_4)a + m_5 \). From the values of \( g(1, 1) \) and \( g(2, 2) \) given, we know that \( m_5 = 3 \) and \( m_3 + m_4 = 2 \). Thus, \( g(20, 20) = 20 \cdot 2 + 3 = 43 \).

An example of a function that works is \( f(a, b) = (a - b)(3a^2 - 3ab + 2a + 3) \). Our sum becomes:

\[
\sum_{a, b \in \mathbb{Z}^+ \atop a \neq b} \frac{1}{2f(a, b)} = \sum_{a, b \in \mathbb{Z}^+ \atop a \neq b} \frac{1}{2(a-b)(3a^2-3ab+2a+3)} = \sum_{a, b \in \mathbb{Z}^+ \atop a \neq b} \frac{1}{2(a-b)^2(3a^2+2a+3)}.
\]

Notice that the summation can be rewritten in terms of \( a \) and \( a + k, k \neq 0 \) instead of \( a \) and \( b \). This gives us:

\[
\sum_{k=1}^{\infty} \sum_{a=1}^{\infty} \frac{1}{2k^2(3a + 2a + 3)\cdot a} + \sum_{k=1}^{\infty} \sum_{a=1}^{\infty} \frac{1}{2k^2(3a - 2a - 3)\cdot a}.
\]

However, since \( 3a + \frac{2a + 3}{k} \geq 3a \) and \( 3a - \frac{2a + 3}{k} \geq a - 3 \), our summation is less than or equal to

\[
\sum_{k=1}^{\infty} \frac{1}{2k^2} \sum_{a=1}^{\infty} \frac{1}{2a} + \sum_{k=1}^{\infty} \frac{1}{2k^2} \sum_{a=1}^{\infty} \frac{1}{2a-3} < 1 \cdot 1 + 1 \cdot 8 = 9.
\]