1. Find the number of pairs (A, B) of distinct subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$, such that A is a proper subset of B.

Answer: 6305

Solution: Since B cannot be empty, the number of elements in B is between 1 and 8. Suppose that B has n elements. There are $2^n - 1$ possible options for A, since A and B are distinct. Thus the total number of pairs is

$$\begin{aligned}\sum_{n=1}^{8}\binom{8}{n}(2^n-1) &= \sum_{n=0}^{8}\binom{8}{n}(2^n-1)\\ &= \sum_{n=0}^{8}\binom{8}{n}2^n - \sum_{n=0}^{8}\binom{8}{n}\\ &= (2+1)^8-(1+1)^8\\ &= \boxed{6305}.\end{aligned}$$

2. What is the remainder when $(5^2+3^2)(5^4+3^4)(5^8+3^8)\dots(5^{2^{419}}+3^{2^{419}})(5^{2^{420}}+3^{2^{420}})$ is divided by 1285?

Answer: 514

Solution 1: Let $S = (5^2+3^2)(5^4+3^4)(5^8+3^8)\dots(5^{2^{419}}+3^{2^{419}})(5^{2^{420}}+3^{2^{420}})$. Then,

$$\begin{aligned}(5^2-3^2)S &= (5^2-3^2)(5^2+3^2)(5^4+3^4)(5^8+3^8)\dots(5^{2^{419}}+3^{2^{419}})(5^{2^{420}}+3^{2^{420}})\\ 16S &= \qquad (5^4-3^4)(5^4+3^4)(5^8+3^8)\dots(5^{2^{419}}+3^{2^{419}})(5^{2^{420}}+3^{2^{420}})\\ &\dots\\ 16S &= 5^{2^{421}}-3^{2^{421}}\\ S &\equiv (16)^{-1}(5^{2^{421}}-3^{2^{421}}) \mod 1285\end{aligned}$$

By Euler's theorem, for relatively prime a and n, $a^{\phi(n)} \equiv 1 \bmod n$. Note that $1285 = 5 \cdot 257$, so $\phi(5) = 4$ and $\phi(257) = 256$ will be helpful. First we consider $S \bmod 257$:

$$\begin{aligned}S &\equiv (16)^{-1}(5^{2^{421} \bmod 256}-3^{2^{421} \bmod 256}) \mod 257\\ S &\equiv (16)^{-1}(1-1) \mod 257\\ S &\equiv 0 \mod 257\end{aligned}$$

Next we consider $S \bmod 5$:

$$\begin{aligned}S &\equiv (16)^{-1}(5^{2^{421}}-3^{2^{421} \bmod 4}) \mod 5\\ S &\equiv (1)(0-1) \mod 5\\ S &\equiv 4 \mod 5\end{aligned}$$

Thus, from $S \equiv 0 \bmod 257$ and $S \equiv 4 \bmod 5$, it follows that $S \equiv \boxed{514} \bmod 1285$.

3. Let $S = \{1, 2, 3, 4, 5\}$. How many ordered pairs of functions (f, g) satisfy $f, g : S \to S$, $f(g(x)) = g(x)$, and $g(f(x)) = f(x)$ for all $x \in S$?

Answer: 1536

Solution: For some $x \in S$, let $g(x) = y$. Then we have $f(g(x)) = g(x) \implies f(y) = y$, and $g(f(x)) = f(x) \implies g(y) = y$. By symmetry, if $f(a) = b$ for some $a \in S$, then $f(b) = b$ and $g(b) = b$.

For a function h, let x be a *fixed point* if $h(x) = x$. Then by above, all fixed points of f must also be fixed points of g and vice versa. Furthermore, if $f(x) = y$ or $g(x) = y$ for $x \neq y$, then y must be a fixed point.

Therefore, we can count the number of ordered pairs of functions by casework over the number of fixed points. If there are n fixed points, there are $\binom{5}{n}$ ways to choose them. For the rest of the $5-n$ elements in the domain, they have to map to a fixed point. Hence, there are n^{5-n} ways to choose the values for the remaining elements in the domain for each of the two functions. As a result, the total number of ordered pairs of functions is

$$\sum_{n=1}^{5} \binom{5}{n} n^{2(5-n)} = \boxed{1536}$$