1. Find the number of pairs \((A, B)\) of distinct subsets of \(\{1, 2, 3, 4, 5, 6, 7, 8\}\), such that \(A\) is a proper subset of \(B\).

Answer: 6305

Solution: Since \(B\) cannot be empty, the number of elements in \(B\) is between 1 and 8. Suppose that \(B\) has \(n\) elements. There are \(2^n - 1\) possible options for \(A\), since \(A\) and \(B\) are distinct. Thus the total number of pairs is

\[
\sum_{n=1}^{8} \binom{8}{n} (2^n - 1) = \sum_{n=0}^{8} \binom{8}{n} (2^n - 1)
\]

\[
= \sum_{n=0}^{8} \binom{8}{n} 2^n - \sum_{n=0}^{8} \binom{8}{n}
\]

\[
= (2 + 1)^8 - (1 + 1)^8 = 6305
\]

2. What is the remainder when \((5^2 + 3^2)(5^4 + 3^4)(5^8 + 3^8)\ldots(5^{2419} + 3^{2419})(5^{2420} + 3^{2420})\) is divided by 1285?

Answer: 514

Solution 1: Let \(S = (5^2 + 3^2)(5^4 + 3^4)(5^8 + 3^8)\ldots(5^{2419} + 3^{2419})(5^{2420} + 3^{2420})\). Then,

\[16S = (5^4 - 3^4)(5^8 + 3^8)\ldots(5^{2419} + 3^{2419})(5^{2420} + 3^{2420})\]

\[\vdots\]

\[16S = 5^{2421} - 3^{2421}\]

\[S \equiv (16)^{-1}(5^{2421} - 3^{2421}) \mod 1285\]

By Euler’s theorem, for relatively prime \(a\) and \(n\), \(a^{\phi(n)} \equiv 1 \mod n\). Note that 1285 = 5 \cdot 257, so \(\phi(5) = 4\) and \(\phi(257) = 256\) will be helpful. First we consider \(S \mod 257\):

\[S \equiv (16)^{-1}(5^{2421} \mod 256 - 3^{2421} \mod 256) \mod 257\]

\[S \equiv (16)^{-1}(1 - 1) \mod 257\]

\[S \equiv 0 \mod 257\]

Next we consider \(S \mod 5\):

\[S \equiv (16)^{-1}(5^{2421} - 3^{2421}) \mod 5\]

\[S \equiv (1)(0 - 1) \mod 5\]

\[S \equiv 4 \mod 5\]

Thus, from \(S \equiv 0 \mod 257\) and \(S \equiv 4 \mod 5\), it follows that \(S \equiv 514 \mod 1285\).

3. Let \(S = \{1, 2, 3, 4, 5\}\). How many ordered pairs of functions \((f, g)\) satisfy \(f, g : S \rightarrow S\), \(f(g(x)) = g(x)\), and \(g(f(x)) = f(x)\) for all \(x \in S\)?

Answer: 1536
**Solution:** For some \( x \in S \), let \( g(x) = y \). Then we have \( f(g(x)) = g(x) \implies f(y) = y \), and \( g(f(x)) = f(x) \implies g(y) = y \). By symmetry, if \( f(a) = b \) for some \( a \in S \), then \( f(b) = b \) and \( g(b) = b \).

For a function \( h \), let \( x \) be a fixed point if \( h(x) = x \). Then by above, all fixed points of \( f \) must also be fixed points of \( g \) and vice versa. Furthermore, if \( f(x) = y \) or \( g(x) = y \) for \( x \neq y \), then \( y \) must be a fixed point.

Therefore, we can count the number of ordered pairs of functions by casework over the number of fixed points. If there are \( n \) fixed points, there are \( \binom{5}{n} \) ways to choose them. For the rest of the \( 5 - n \) elements in the domain, they have to map to a fixed point. Hence, there are \( n^{5-n} \) ways to choose the values for the remaining elements in the domain for each of the two functions. As a result, the total number of ordered pairs of functions is

\[
\sum_{n=1}^{5} \binom{5}{n} n^{2(5-n)} = 1536
\]