1. Consider a semi-circle with diameter $AB$. Let points $C$ and $D$ be on diameter $AB$ such that $CD$ forms the base of a square inscribed in the semicircle. Given that $CD = 2$, compute the length of $AB$.

**Answer:** $2\sqrt{5}$

**Solution:** Note that the center of the semi-circle lies on the center of one of the sides of the square. If we draw a line from the center to an opposite corner of the square, we form a right triangle whose side lengths are 1 and 2 and whose hypotenuse is the radius of the semicircle. We can therefore use the Pythagorean Theorem to compute $r = \sqrt{1^2 + 2^2} = \sqrt{5}$. The radius is half the length of $AB$ so therefore $AB = \left[2\sqrt{5}\right]$. 

2. Let $ABCD$ be a trapezoid with $AB$ parallel to $CD$ and perpendicular to $BC$. Let $M$ be a point on $BC$ such that $\angle AMB = \angle DMC$. If $AB = 3$, $BC = 24$, and $CD = 4$, what is the value of $AM + MD$?

**Answer:** 25

**Solution:** Let $A'$ be the reflection of $A$ by $BC$. We have $\angle A'MB = \angle AMB = \angle DMC$. Hence, $A', M$, and $D$ are collinear. Let $C'$ be the intersection of the line parallel to $BC$ passing through $A'$ and the extension of $DC$. We have $\angle A'C'D = 90^\circ$, $A'C' = BC = 24$, and $C'D = C'C + CD = A'B + CD = AB + CD = 3 + 4 = 7$. Therefore, $AM + MD = A'M + MD = A'D = \sqrt{A'C'^2 + C'D^2} = \sqrt{24^2 + 7^2} = \left[25\right]$ by Pythagorean Theorem.

3. Let $ABC$ be a triangle and $D$ be a point such that $A$ and $D$ are on opposite sides of $BC$. Give that $\angle ACD = 75^\circ$, $AC = 2$, $BD = \sqrt{6}$, and $AD$ is an angle bisector of both $\triangle ABC$ and $\triangle BCD$, find the area of quadrilateral $ABDC$.

**Answer:** $3 + \sqrt{3}$

**Solution 1:** Since $AD$ is an angle bisector of both $\triangle ABC$ and $\triangle BCD$, $\angle BAD = \angle CAD$ and $\angle BDA = \angle CDA$. Then by angle-side-angle congruence, $\triangle ABD \cong \triangle ACD$, and $CD = BD = \sqrt{6}$. Since $\angle ACD = 75^\circ$, we can use the area formula

$$[ACD] = \frac{1}{2} AC \cdot CD \cdot \sin 75^\circ = \frac{1}{2} \cdot 2 \cdot \sqrt{6} \cdot \frac{\sqrt{2} + \sqrt{6}}{4} = \frac{\sqrt{3} + 3}{2}.$$ 

Because $[ABD] = [ACD]$ we have $[ABCD] = 2[ACD] = \left[3 + \sqrt{3}\right]$. 

**Solution 2:** We begin as in the original solution by noticing $\triangle ABD \cong \triangle ACD$. This implies that $ABCD$ is a kite, which means that $AD \perp BC$. Let $E$ be the intersection of $AD$ and $BC$. Note that then $\triangle CAE$ is a $30 - 60 - 90$ right triangle, while $\triangle CDE$ is a $45 - 45 - 90$ right triangle. This gives us $AD = 1 + \sqrt{3}$ and $BC = 2\sqrt{3}$, so the area of kite $ABCD$ is $\frac{1}{2} AD \cdot BC = \left[3 + \sqrt{3}\right]$.

4. Let $a_1, a_2, ..., a_{12}$ be the vertices of a regular dodecagon $D_1$ (12-gon). The four vertices $a_1, a_4, a_7, a_{10}$ form a square, as do the four vertices $a_2, a_5, a_8, a_{11}$ and $a_3, a_6, a_9, a_{12}$. Let $D_2$ be the polygon formed by the intersection of these three squares. If we let $[A]$ denotes the area of polygon $A$, compute $\frac{[D_2]}{[D_1]}$.

**Answer:** $4 - 2\sqrt{3}$

**Solution:** By symmetry, $D_2$ is also a regular dodecagon. Therefore, to find the ratio of areas, we need only find the ratio of side lengths between the two dodecagons.

We begin by labeling relevant points and lines. Let $X$ be the intersection of $a_1a_3$ and $a_2a_5$; $Y$ be the intersection of $a_1a_3$ and $a_1a_4$; and $Z$ be the intersection of $a_1a_4$ and $a_2a_5$. Then $YZ$ is a side length of $D_2$, so we must find the ratio $YZ/a_2a_3$. Note that $a_1a_4$ is parallel to $a_2a_3$, so $XY$ is also parallel to $a_2a_3$, which implies that $\triangle a_2a_3X \sim \triangle ZYX$. Thus, $\frac{YZ}{a_2a_3} = \frac{XZ}{a_2X}$. 


Let $x$ denote the length of $a_2a_3$. Since $D_1$ is a 12-gon, each angle is $\frac{10\cdot 180}{12} = 150$ degrees. We know that $\angle a_1a_2a_3$ is the right angle of a square, and by symmetry $\angle a_1a_2a_4 = \angle a_2a_3a_3$, so $\angle Xa_2a_3 = \frac{150-90}{2} = 30$. Again, by symmetry we also have $\angle Xa_3a_2 = 30$. Using $30-60-90$ right triangles, we see that $a_2X = x/\sqrt{3}$. On the other hand, since $\angle Za_1a_2 = 30$ by symmetry and $\angle a_1Za_2 = \angle YZX = 30$ by similar triangles, we see that $\angle a_1Za_2$ is isosceles, and therefore $a_2Z = x$. Since $XZ = a_2Z - a_2X$, we have $XZ = x - x/\sqrt{3}$. Therefore, $\frac{XZ}{a_2X} = \frac{x-x/\sqrt{3}}{x/\sqrt{3}} = \sqrt{3} - 1$. Squaring this gives us the ratio of areas $\frac{Xa_2}{Xa_3} = \frac{4 - 2\sqrt{3}}{1}$.

5. In $\triangle ABC$, $\angle ABC = 75^\circ$ and $\angle BAC$ is obtuse. Points $D$ and $E$ are on $AC$ and $BC$, respectively, such that $\frac{AB}{BC} = \frac{DE}{EC}$ and $\angle DEC = \angle EDC$. Compute $\angle DEC$ in degrees.

**Answer:** $85$

**Solution:** Extend $AC$ past $A$, and draw $F$ on $AC$ such that $AB = FB$. Note that $\frac{AB}{BC} = \frac{FB}{EC}$, and since $\angle FBC = \angle DEC$ we have $\triangle FBC \sim \triangle DEC$.

Next, we perform some angle chasing. Let $\angle DEC = \angle EDC = x$. By similar triangles, we have $\angle FBC = \angle BFC = x$ as well. Furthermore, $FB = AB$, so $\triangle FAB$ is isosceles, and thus $\angle FAB = x$ as well. Now $\angle ABC = 75^\circ$, so we compute $\angle FBA = \angle FBC - \angle ABC = x - 75$. The angles of a triangle sum to $180^\circ$, giving us the equation $3x - 75 = 180$, which solves to $x = 85$.

6. In $\triangle ABC$, $AB = 3$, $AC = 6$, and $D$ is drawn on $BC$ such that $AD$ is the angle bisector of $\angle BAC$. $D$ is reflected across $AB$ to a point $E$, and suppose that $AC$ and $BE$ are parallel. Compute $CE$.

**Answer:** $\sqrt{61}$

**Solution:** Let $\angle BAC = x$. Since $AC \parallel BE$, we have $\angle ABE = x$, so $\angle ABC = x$ since $E$ is the reflection of $D$ across $AB$. This means that $\triangle ABC$ is isosceles, so $AC = BC = 6$. Using the angle bisector theorem, we find $CD = 4$ and $BD = BE = 2$.

Now let $\theta = \angle CBA$. We can use the Law of Cosines to compute

$$CE = \sqrt{BC^2 + BE^2 - 2 \cdot BC \cdot BE \cdot \cos 2\theta}.$$ 

Because $\triangle ABC$ is isosceles, we can drop the perpendicular from $C$ to find $\cos \theta = \frac{15}{6} = \frac{5}{1}$. Using the double angle formula, we get $\cos 2\theta = 2\cos^2 \theta - 1 = -\frac{7}{8}$. Plugging this in gives us the answer

$$CE = \sqrt{2^2 + 6^2 - 2 \cdot 2 \cdot 6 \cdot \left(-\frac{7}{8}\right)} = \sqrt{61}.$$

7. Two equilateral triangles $ABC$ and $DEF$, each with side length 1, are drawn in 2 parallel planes such that when one plane is projected onto the other, the vertices of the triangles form a regular hexagon $AFBDCE$. Line segments $AE$, $AF$, $BF$, $BD$, $CD$, and $CE$ are drawn, and suppose that each of these segments also has length 1. Compute the volume of the resulting solid that is formed.

**Answer:** $\frac{\sqrt{2}}{3}$

**Solution 1:** Draw lines $l_1, l_2, l_3$ through $A$ parallel to $BC$, through $B$ parallel to $CA$, and through $C$ parallel to $AB$, respectively. Suppose that $l_2$ and $l_3$ intersect at $D'$, $l_3$ and $l_1$ intersect at $E'$, and $l_1$ and $l_2$ intersect at $F'$. We observe that $DD', EE'$, and $FF'$ are concurrent at a single point $G$. In addition, $\triangle D'E'F'$ is an equilateral triangle with side length 2, double that of $\triangle DEF$.

Let the notation $[f]$ denote the volume of a figure $f$. Then, we observe that $[D'E'F'G] = [ABCDEF] + [ABFF'] + [BCDD'] + [CAEE'] + [DEFG]$. Next, we wish to show that
Let $D'E'F'G$ be regular tetrahedra with side length 1. Then $FF' 
eq 1$ as well, since $EF || E'F'$. Since $AE = AE' = AF = AF' = 1$, both $AEE'$ and $AFF'$ are not equilateral, and $\angle EAF = \angle FAF'$ is not $60^\circ$. However, this means that $\angle EAF \neq 60^\circ$ since $\angle EAE + \angle EAF + \angle FAF' = 180^\circ$, contradicting the fact that $\angle EAF$ is equilateral with side length 1.

Hence, we have $DD' = EE' = FF'$, after extrapolating the previous argument to the third side. Therefore, $[ABCD] = [D'E'F'G] - 4[DEFG] = 8[DEFG] - 4[DEFG] = 4[DEFG]$, since a volume which is scaled by twice the side length has its volume scaled by $2^3 = 8$. It suffices to compute the volume of a regular tetrahedron. Dropping the altitude from $G$ to $\triangle DEF$, we can compute the height of the tetrahedron $DEFG$ to be $\frac{\sqrt{2}}{3}$. Hence, the volume of $DEFG$ is

$$\frac{1}{3} \cdot \frac{\sqrt{6}}{3} \cdot A[\triangle DEF] = \frac{\sqrt{6}}{9} \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{2}}{12}.$$ 

Finally, our desired area is $4 \times \frac{\sqrt{2}}{12} = \frac{\sqrt{2}}{3}$.

**Solution 2:** Because each of the edges of the solid have length 1, each of the faces of the solid are equilateral triangles. There are 12 edges, which implies that the solid is in fact a regular octahedron with edge length 1.

To compute the volume of a regular octahedron, we split it into two congruent square pyramids. Let $ABCD$ be the square base with side length 1, and let $E$ be the top of the pyramid. Also, let $F$ be the point in the middle of $ABCD$ and let $M$ be the midpoint of $AB$. Clearly the base $ABCD$ has area 1. To compute the height $EF$, we note that $\triangle EFM$ is a right triangle with $FM = \frac{1}{2}$. To compute $EM$, we note that it is a leg of the right triangle $\triangle AME$ where $AM = \frac{1}{2}$ and $AE = 1$. Using the Pythagorean Theorem on $\triangle AME$ gives us $EM = \frac{\sqrt{3}}{2}$, and using it again on $\triangle EFM$ gives us the height $EF = \frac{\sqrt{2}}{2}$. Therefore, the volume of the square pyramid $ABCD$ is $\frac{1}{3} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6}$, which implies that the volume of the regular octahedron is $\frac{\sqrt{2}}{3}$.

8. Let $ABC$ be a right triangle with $\angle ACB = 90^\circ$, $BC = 16$, and $AC = 12$. Let the angle bisectors of $\angle BAC$ and $\angle ABC$ intersect $BC$ and $AC$ at $D$ and $E$ respectively. Let $AD$ and $BE$ intersect at $I$, and let the circle centered at $I$ passing through $C$ intersect $AB$ at $P$ and $Q$ such that $AQ < AP$. Compute the area of quadrilateral $DPQE$.

**Answer:** $\frac{136}{3}$

**Solution:** We claim that $AP = AC$. Let $P'$ be a point on $AB$ such that $AP' = AC$. Since $AI = AI$ and $\angle CAI = \angle P'AI$, we have $\triangle ACI \cong \triangle AP'I$ by SAS congruency. Thus, $IC = IP'$, so $P'$ must lie on the circle centered at $I$ passing through $C$. This implies that either $P = P'$ or $Q = P'$.

Let the circle centered at $I$ passing through $C$ intersect $AC$ at $X$. Note that $AX < AC$, and by Power of a Point, $AX \cdot AC = AQ \cdot AP$.

Now suppose that $P' = Q$. Then $AC = AP' = AQ$, so $AP = AX < AC = AQ$, contradicting $AQ < AP$. Hence, $P' \neq Q$, so $P'$ is $P$ and $AP = AC$. 


From here, we notice that $\angle PAD = \angle CAD$ and $AD = AD$, so $\triangle ADP \cong \triangle ADC$ by SAS congruency. As a result, $CD = PD$ and $\angle APD = 90^\circ$. By similar reasoning, we deduce that $CE = QE$ and $\angle BQE = 90^\circ$. It follows that $DPQE$ is a trapezoid. By the Angle Bisector Theorem, we compute that 

$$PD = CD = \frac{AC}{AB + AC} \cdot BC = \frac{12}{32} \cdot 16 = 6.$$ 

Likewise, we compute 

$$QE = CE = \frac{BC}{BC + AB} \cdot AC = \frac{16}{36} \cdot 12 = \frac{16}{3}$$

Finally, we have 

$$PQ = AP + BQ - AB = AC + BC - AB = 8$$

so the area of $DPQE$ is 

$$\frac{1}{2}PQ(QE + PD) = \frac{1}{2} \cdot 8 \left( \frac{16}{3} + 6 \right) = \frac{136}{3}.$$ 

9. Let $ABCD$ be a cyclic quadrilateral with $3AB = 2AD$ and $BC = CD$. The diagonals $AC$ and $BD$ intersect at point $X$. Let $E$ be a point on $AD$ such that $DE = AB$ and $Y$ be the point of intersection of lines $AC$ and $BE$. If the area of triangle $ABY$ is 5, then what is the area of quadrilateral $DEYX$? 

**Answer:** 11

**Solution 1:** Let $[A]$ denote the area of polygon $A$. Since $BC = CD$, $\angle BDC = \angle DBC$. Note that $\angle BAC = \angle BDC$ since they are angles of the same segment and $\angle CAD = \angle CBD$ for the same reason. Hence, $\angle BAC = \angle CAD$ and thus $AC$ is the angle bisector of $\angle BAD$. Therefore, 

$$\frac{BX}{XD} = \frac{AB}{AD} = \frac{2}{3}.$$ 

We also know that 

$$\frac{DE}{EA} = \frac{DE}{AD - DE} = \frac{AB}{AD - AB} = 2.$$ 

Now by Menelaus’ Theorem, we have 

$$\frac{AY}{YX} \cdot \frac{X B}{BD} \cdot \frac{DE}{EA} = 1.$$ 

Therefore, $\frac{AY}{YX} = \frac{5}{4}$ so $[BXY] = 4$. Since $\frac{BX}{XD} = \frac{2}{3}$, we have $[DXY] = \frac{3}{2} \cdot 4 = 6$. Now since $AY$ is the angle bisector of $\angle BAE$, we have 

$$\frac{BY}{YE} = \frac{AB}{AE} = 2$$

and thus $[AEY] = \frac{5}{2}$. Because $\frac{DE}{EA} = 2$, we have $[DEY] = 2[AEY] = 5$. Finally, we have $[DEYX] = [DEY] + [DXY] = 5 + 6 = 11$

**Solution 2:** Instead of using Menelaus’ Theorem, let the $x = [BXY]$ and $y = [DEYX]$. We know that $[AEY] = \frac{5}{2}$ from above. We then have the two equations 

$$\frac{5 + x}{\frac{5}{2} + y} = \frac{2}{3} \quad \frac{5 + \frac{5}{2}}{x + y} = \frac{1}{2}.$$ 

Solving these two equations gives us $x = 4$ and $y = \frac{11}{2}$. 


10. Let $ABC$ be a triangle with $AB = 13$, $AC = 14$, and $BC = 15$, and let $\Gamma$ be its incircle with incenter $I$. Let $D$ and $E$ be the points of tangency between $\Gamma$ and $BC$ and $AC$ respectively, and let $\omega$ be the circle inscribed in $CDIE$. If $Q$ is the intersection point between $\Gamma$ and $\omega$ and $P$ is the intersection point between $CQ$ and $\omega$, compute the length of $PQ$.

**Answer:** $8\sqrt{\frac{6}{9}}$

**Solution:** We can derive that $CD = CE = 8$. We then compute the inradius $r$ of $\triangle ABC$. Using Heron’s Formula or drawing an altitude from $B$ to $AC$, we can calculate that the area of $\triangle ABC$ is 84. Since the product of $r$ and the semiperimeter of $\triangle ABC$ also gives the area, we find that $r = 4$.

Let $O$ be the center of $\omega$. Also let $\omega$ touch $ID$ at $X$ and $CD$ at $Y$. Since $OXDY$ is a square, we have $\triangle IYO \cong \triangle OYC$. Let $x$ be the radius of $\omega$, giving us $XO = YO = x$ and $IX = 4 - x$ and $CY = 8 - x$. This gives us the equation $\frac{4-x}{x} = \frac{x}{8-x}$, and solving for $x$ yields $x = \frac{8}{3}$.

Since both $\omega$ and $\Gamma$ are tangent to $AC$ and $BC$, a homothety (the enlargement/shrinking of objects with respect to a fixed point and fixed ratio) centered at $C$ sends $\omega$ to $\Gamma$, and the ratio is $\frac{x}{r} = \frac{2}{3}$. Since $C, P, Q$ are collinear, the same homothety also takes $P$ to $Q$, so $\frac{CP}{CQ} = \frac{2}{3}$. Letting $CP = 2k$ and $CQ = 3k$, we have that $PQ = k$. Finally, $CY = 8 - \frac{8}{3} = \frac{16}{3}$, so by Power of a Point,

$$CY^2 = CP \cdot CQ \implies \left(\frac{16}{3}\right)^2 = 6k^2.$$

Solving for $k$ gives us $PQ = \frac{8\sqrt{6}}{9}$. 
