1. A number is formed using the digits $\{2, 0, 1, 8\}$, using all 4 digits exactly once. Note that $0218 = 218$ is a valid number that can be formed. What is the probability that the resulting number is strictly greater than 2018?

   **Answer:** $\frac{11}{24}$

   **Solution:** There are $4! = 24$ total possible permutations of the digits $\{2, 0, 1, 8\}$. Next, we count the number of cases where the permutation is greater than 2018. When the leftmost digit is an 8, the permutation must be greater than 2018, and there are $3! = 6$ such permutations. When the leftmost digit is a 2, the permutation is not greater than 2018 only when the permutation is 2018 itself, since $0 < 1 < 8$. In this case, there are $3! - 1 = 5$ such permutations. Because all other permutations of the digits have a thousands digit which is less than 2, the desired probability is $\frac{6+5}{24} = \boxed{\frac{11}{24}}$.

2. Let $S(n)$ denote the sum of the digits of the integer $n$. If $S(n) = 2018$, what is the smallest possible value $S(n + 1)$ can be?

   **Answer:** 3

   **Solution:** To minimize $S(n + 1)$, when we add 1 to $n$, we should change as many digits possible from 9 to 0 due to the carry. For example, if we have $n = 199$, adding 1 gives us $n + 1 = 200$, changing 2 nines to zeros. Since $2018 = 224 \cdot 9 + 2$, we can let $n$ be a number starting with the digit 2 followed by 224 nines. Then adding 1 to $n$ gives us the digit 3 followed by 224 zeros. Hence, $S(n + 1) = \boxed{3}$.

3. A dice is labeled with the integers $1, 2, ..., n$ such that it is 2 times as likely to roll a 2 as it is a 1, 3 times as likely to roll a 3 as it is a 1, and so on. Suppose the probability of rolling an odd integer with the dice is $\frac{17}{35}$. Compute $n$.

   **Answer:** 34

   **Solution:** We first observe that the dice must have an even number of sides. Suppose otherwise. Then the probability of rolling an $n$, an odd number, is higher than that of rolling $n - 1$; the probability of rolling $n - 2$ is higher than that of rolling $n - 3$, and so on. This would imply that the probability of rolling an odd number is greater than the probability of rolling an even number and thus greater than 0.5, a contradiction.

   Let $n = 2k$. Then, the probability of rolling an odd number is

$$\begin{aligned}\frac{17}{35} &= \frac{1 + 3 + 5 + \cdots + 2k - 1}{1 + 2 + 3 + \cdots + 2k} \\ &= \frac{k^2}{2k(2k + 1)/2} \\ &= \frac{k}{2k + 1}\end{aligned}$$

   Hence, we must have $k = 17$, and $n = 2k = \boxed{34}$.

4. One of the six digits in the expression $435 \cdot 605$ can be changed so that the product is a perfect square $N^2$. Compute $N$.

   **Answer:** 495

   **Solution:** We first factorize $435 = 3 \cdot 5 \cdot 29$ and $605 = 5 \cdot 11^2$. First, suppose that we change 605. In this case, we could change it to be of the form $(3 \cdot 5 \cdot 29)k^2$ for some positive integer $k$. If $k = 1$, we clearly cannot change 605 to 435 by only modifying a single digit, and if $k \geq 2$, then this number would have more than 4 digits.

Therefore, we must change 435 so that it is of the form $5k^2$. It must be divisible by 5, so the last digit must be 0 or 5. If we change the last digit to a 0, then we have $430 = 5k^2 \implies k^2 = 86$, which is not a perfect square. Therefore, the last digit must remain a 5, which also implies that $k$ must be odd. Computing $5k^2$ for small odd values of $k$ gives us the three digit numbers 125, 245, 405, 605, and 845. The only number that can be formed by changing a single digit of 435 is $405 = 3^4 \cdot 5$, so we have $N^2 = 405 \cdot 605 = 3^4 \cdot 5^2 \cdot 11^2$. Taking the square root gives us the answer $N = 3^2 \cdot 5 \cdot 11 = \boxed{495}$.

5. A sequence is defined as follows. Given a term $a_n$, we define the next term $a_{n+1}$ as

$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even} \\ a_n - 1 & \text{if } a_n \text{ is odd} \end{cases}$$

The sequence terminates when $a_n = 1$. Let $P(x)$ be the number of terms in such a sequence with initial term $x$. For example, $P(7) = 5$ because its corresponding sequence is $7, 6, 3, 2, 1$. Evaluate $P(2^{2018} - 2018)$.

**Answer: 4028**

**Solution:** We convert the number into its binary equivalent. If the number ends with 0 (which means it is even), the next term has the same binary form with the 0 removed. If the number ends with 1 (which means it is odd), the next term has the same binary form with the last digit changed to 0. Given a number $x$ with a binary representation $p$, we compute $P(x)$ by summing over the digits in $p$ and obtain

$$P(x) = (\text{\# of digits in } p) + (\text{\# of 1's in } p) - 1.$$

The binary representation

$$2^{2018} - 2018 = 2^{2018} - 1 - 2017 = 2^{2018} - 1 - (1 + 32 + 64 + 128 + 256 + 512 + 1024)$$

consists of 2011 1's and 7 0's, so $P(2018) = 2018 + 2011 - 1 = \boxed{4028}$.

6. Elizabeth is at a candy store buying jelly beans. Elizabeth begins with 0 jellybeans. With each scoop, she can increase her jellybean count to the next largest multiple of 30, 70 or 110. (For example, her next scoop after 70 can increase her jellybean count to 90, 110, or 140). What is the smallest number of jellybeans Elizabeth can collect in more than 100 different ways?

**Answer: 210**

**Solution:** Let $J(n)$ be the number of different ways to collect $n$ jellybeans. Note that $J(30k)$ is the sum of all $J(n)$ where $30(k-1) \le n < 30k$. A similar thing is true for $J(70k)$ and $J(110k)$. We can then compute the following values for $J(n)$:

$$\begin{aligned}
J(0) &= 1 \\
J(30) &= 1 \\
J(60) &= 1 \\
J(70) &= J(0) + J(30) + J(60) = 3 \\
J(90) &= J(60) + J(70) = 4 \\
J(110) &= J(0) + J(30) + J(60) + J(70) + J(90) = 10 \\
J(120) &= J(90) + J(110) = 14 \\
J(140) &= J(70) + J(90) + J(110) + J(120) = 31 \\
J(150) &= J(120) + J(140) = 45 \\
J(180) &= J(150) = 45 \\
J(210) &= J(140) + J(150) + J(180) > 100
\end{aligned}$$

The smallest number is thus $\boxed{210}$.

7. Let $S$ be the set of all 1000 element subsets of the set $\{1, 2, 3, ..., 2018\}$. What is the expected value of the minimum element of a set chosen uniformly at random from $S$?

**Answer:** $\mathbf{\frac{2019}{1001}}$

**Solution:** Note that the minimum element of a subset of size 1000 ranges from 1 to 1019 inclusive. Therefore, for every $k$ satisfying $1 \le k \le 1019$, we count how many times it is the minimum value in a subset. From the set $\{k+1, k+2, ..., 2018\}$, we must choose 999 additional elements, so there are $\binom{2018-k}{999}$ such subsets. Since there are $\binom{2018}{1000}$ sets in $S$, our desired expected value is

$$\frac{\binom{2017}{999} + 2\binom{2016}{999} + 3\binom{2015}{999} + \cdots + 1019\binom{999}{999}}{\binom{2018}{1000}}.$$

The numerator can be rewritten as

$$\begin{gathered}\binom{2017}{999} + \binom{2016}{999} + \binom{2015}{999} + \cdots + \binom{999}{999} \\ + \\ \binom{2016}{999} + \binom{2015}{999} + \cdots + \binom{999}{999} \\ + \\ \vdots \\ + \\ \binom{999}{999}\end{gathered}$$

Applying the hockey-stick identity to each line, the expression becomes

$$\binom{2018}{1000} + \binom{2017}{1000} + \cdots + \binom{1000}{1000}.$$

Using the hockey-stick identity once more, we obtain $\binom{2019}{1001}$. Thus, the expected value is

$$\frac{\binom{2019}{1001}}{\binom{2018}{1000}} = \frac{\frac{2019!}{1001!\cdot 1018!}}{\frac{2018!}{1000!\cdot 1018!}} = \boxed{\frac{2019}{1001}}.$$

8. Positive integer $n$ can be written in the form $a^2 - b^2$ for at least 12 pairs of positive integers $(a, b)$. Compute the smallest possible value of $n$.

**Answer: 1440**

**Solution:** Since $n = a^2 - b^2 = (a-b)(a+b)$, the values $a-b$ and $a+b$ must have the same parity, otherwise this would imply that both $a$ and $b$ are not integral. Therefore, if $n$ is divisible by 2, then both $a-b$ and $a+b$ must be divisible by 2. It suffices to compute the number of factor pairs $(a-b, a+b)$ of $n$ such that $a-b$ and $a+b$ share the same parity, since $(a, b)$ uniquely determines $(a-b, a+b)$ and vice versa.

First, suppose that $n$ is divisible by 2. Let $n = 2^{p_2}3^{p_3}5^{p_5}\cdots$ for positive $p_2$ and some nonnegative integers $p_3, p_5, ....$ Then, the number of valid factor pairs of $n$ is

$$\left\lfloor \frac{(p_2 - 1)(p_3 + 1)(p_5 + 1)\cdots}{2} \right\rfloor = 12.$$

Here, we take $p_2 - 1$ to account for the equivalent parity of the factor pair, as the exponent of 2 in either integer of the factor pair cannot be 0 or $p_2$. Next, we divide by 2 and take the floor to take into account only valid factor pairs, as opposed to just factors of $n$.

There are two cases:

(a) $(p_2 - 1)(p_3 + 1)(p_5 + 1)\cdots = 24$. After doing some case work, we find that setting $p_2 - 1 = 4$, $p_3 + 1 = 3$, and $p_5 + 1 = 2$ yields the smallest possible value of $n$, which is $2^5 3^2 5^1 = 1440$.

(b) $(p_2 - 1)(p_3 + 1)(p_5 + 1)\cdots = 25$. After doing some case work, we find that setting $p_2 - 1 = 5$ and $p_3 + 1 = 5$ yields the smallest possible value of $n$, which is $2^6 3^4 = 5184$.

We can repeat the process with $n$ not divisible by 2, and the number of valid factor pairs of $n$ is instead

$$\left\lfloor \frac{(p_3 + 1)(p_5 + 1)...}{2} \right\rfloor = 12$$

where we do not need to take into account parity since all such $n$ not divisible by 2 are odd. However, we quickly observe that $n$ will be extremely large if it is not divisible by 2, and we can ignore this case.

Hence, the smallest possible value of $n$ is $\boxed{1440}$.

9. Let

$$S = \sum_{k=1}^{2018102} \sum_{n=1}^{1008} n^k.$$

Compute the remainder when $S$ is divided by 1009.

**Answer: 16**

**Solution:** Note that $p = 1009$ is a prime. We claim that for any fixed prime $p$ and integer $k$, the sum

$$\sum_{n=1}^{p-1} n^k \pmod p$$

is $-1 \mod p$ if $(p-1) \mid k$ and $0 \mod p$ if $(p-1) \nmid k$.

We first consider the case where $(p-1) \mid k$. By Fermat's little theorem, $n^k \equiv 1 \pmod p$ for all $n = 1, ..., p-1$. Therefore, we have

$$\sum_{n=1}^{p-1} n^k \equiv -1 \pmod p$$

in this case.

Now consider the case where $(p-1) \nmid k$. Because $p$ is prime, there exists some generator $g$ such that $g^{p-1} \equiv 1 \pmod p$ but $g^a \not\equiv 1 \pmod p$ for all $0 < a < p-1$. This implies that for all $1 \le n \le p-1$, there exists a unique $1 \le a \le p-1$ such that $n \equiv g^a \pmod p$. Otherwise, if there exists two different $a, b$ such that $g^a \equiv g^b \equiv n \pmod p$, we would have $g^{a-b} \equiv 1 \pmod p$, which implies that $a - b = 0$, or $a = b$. Therefore, we can rewrite the sum as

$$\sum_{n=1}^{p-1} n^k \equiv \sum_{n=1}^{p-1} g^{nk} \pmod p.$$

However, if we multiply the sum by $1 - g^k$, we find that the sum telescopes, and we get

$$(1 - g^k) \sum_{n=1}^{p-1} g^{nk} \equiv g^k - g^{pk} \equiv g^k(1 - (g^{p-1})^k) \equiv 0 \pmod p$$

where $g^{p-1} \equiv 1 \pmod{p}$ from Fermat's Little Theorem. However, because $k \nmid p-1$, we know that $g^k \not\equiv 1 \pmod{p}$, which implies that

$$\sum_{n=1}^{p-1} n^k \equiv 0 \pmod{p}$$

in this case.

Thus, we have

$$S \equiv -\left\lfloor \frac{2018102}{1008} \right\rfloor \equiv -2002 \equiv \boxed{16} \pmod{p}.$$

10. Morris plays a game using a fair coin. He starts with \$2 and proceeds using the following rules:

- If Morris flips a heads, he gains \$2.
- If Morris flips a tails, he loses half his money.
- If Morris flips two tails in a row, the game ends (but he doesn't lose any more money)

For instance, if Morris flips the sequence THTT, he will end up with \$1.50. What is the expected amount of money in dollars Morris will have after the game ends?

**Answer:** $\frac{8}{3}$

**Solution:** Let $f(x)$ be the expected amount of money Morris has at the end of the game if he starts with \$$x$. We split the original game into two slightly different games, where the winnings of the original game is the sum of the winnings of the two other games.

Game 1: Morris starts the game with \$$x$.

- If Morris flips a heads, nothing happens.
- If Morris flips a tails, he loses half his money.
- If Morris flips two tails in a row, the game ends (but he doesn't lose any more money)

Game 2: Morris starts the game with \$0.

- If Morris flips a heads, he gains \$2.
- If Morris flips a tails, he loses half his money.
- If Morris flips two tails in a row, the game ends (but he doesn't lose any more money)

Intuitively, the first game represents the decay of our original investment of \$2 by flipping tails, while the second game represents our winnings when we start with \$0.

We first compute the expected value of game 1. The probability that we halve exactly once is $\frac{1}{2}$, since that means that we flipped another tails immediately after flipping our first one. Likewise, the probability that we halve exactly twice is $\frac{1}{4}$, and so on, so the probability that we exactly halve $k$ times is $\frac{1}{2^k}$. The expected value is therefore

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{x}{2^k} = \sum_{k=1}^{\infty} \frac{x}{4^k} = \frac{x}{3}.$$

By definition, the expected value of game 2 is $f(0)$. Thus, the expected value of original the game when we start with \$$x$ is $f(x) = \frac{x}{3} + f(0)$. Plugging in $x = 2$ gives us the equation $f(2) = \frac{2}{3} + f(0)$. On the other hand, consider the case when we start with \$0. We have a $\frac{1}{2}$ chance of flipping heads and gaining \$2, which is the same as starting the game again with \$2. We have a $\frac{1}{4}$ chance of flipping tails and then heads, which is equivalent to the scenario above. Finally we have a $\frac{1}{4}$ chance of flipping tails twice in a row and ending the game. This gives us the equation $f(0) = \frac{1}{2}f(2) + \frac{1}{4}f(2) + \frac{1}{4} \cdot 0$. Plugging this into the other equation and solving, we get $f(2) = \boxed{\frac{8}{3}}$.