1. If \( f(x) = x^2 \) and \( g(x) = \ln(x) \), compute \( f'(1) + g'(1) \).

**Answer:** 3

**Solution:** We compute \( f'(x) = 2x \) and \( g'(x) = \frac{1}{x} \), so plugging in 1 for both gives us 3.

2. Given that \( f(x) = x^2 + ax - 17 \), find all real values of \( a \) such that \( f(4) = f'(4) \).

**Answer:** 3

**Solution:** Differentiating \( f(x) \) gives us \( f'(x) = 2x + a \). Now \( f(4) = 4a - 1 \) and \( f'(4) = a + 8 \), so setting the two equal yields \( 4a - 1 = a + 8 \), which solves to \( a = 3 \).

3. Find the value of \( a \) such that \( \int_1^a (3x^2 - 6x + 3) \, dx = 27 \).

**Answer:** 4

**Solution:** We factor the integrand to get
\[
\int_1^a 3(x-1)^2 \, dx = (x-1)^3 \bigg|_1^a = (a-1)^3 = 27.
\]
Taking the cube root of both sides yields \( a - 1 = 3 \) or \( a = 4 \).

4. Compute
\[ \int_0^4 \frac{dx}{\sqrt{|x-2|}}. \]

**Answer:** \( 4\sqrt{2} \)

**Solution:** Note that the function \( \frac{1}{\sqrt{|x-2|}} \) is discontinuous at \( x = 2 \). We therefore split the integral into two parts and compute separately:
\[
\int_0^2 \frac{dx}{\sqrt{2-x}} = \int_0^2 \frac{dx}{\sqrt{2-x}} = -2\sqrt{2-x}\bigg|_0^2 = 2\sqrt{2}
\]
\[
\int_2^4 \frac{dx}{\sqrt{x-2}} = \int_2^4 \frac{dx}{\sqrt{x-2}} = 2\sqrt{x-2}\bigg|_2^4 = 2\sqrt{2}
\]
The answer is therefore \( 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2} \).

5. Eric and Harrison are standing in a field, and Eric is 400 feet directly East of Harrison. Eric starts to walk North at a rate of 4 feet per second, while Harrsion starts to walk South at the same time at a rate of 6 feet per second. After 30 seconds, at what rate is the distance between Eric and Harrison changing?

**Answer:** 6

**Solution:** We can model the rate of a change as a right triangle with base \( x = 400 \) feet and height \( y \) increasing at a rate of 10 feet per second. After 30 seconds, we will have \( y = 300 \) feet. If we let \( z \) denote the distance between Eric and Harrison, then \( z \) is the hypotenuse of the right triangle, giving us the relation \( z^2 = x^2 + y^2 \). Thus, the distance between Eric and Harrison after 30 seconds is \( z = 500 \) feet. Differentiating both sides also yields \( 2z \cdot z' = 2x \cdot x' + 2y \cdot y' \). Plugging in our values, we get \( 2 \cdot 500z' = 2 \cdot 400 \cdot 0 + 2 \cdot 300 \cdot 10 \), which gives us \( z' = 6 \) feet per second.
6. Compute

\[ \lim_{x \to 0} \frac{(1 - \cos x)^2}{x^2 - x^2 \cos^2 x}. \]

**Answer:** \( \frac{1}{4} \)

**Solution:** Factoring out the \( x^2 \) from the denominator and recalling that \( \sin^2 x + \cos^2 x = 1 \), our limit becomes

\[ \lim_{x \to 0} \frac{(1 - \cos x)^2}{x^2 \sin^2 x} = \left( \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} \right)^2. \]

Note that the limit is in an indeterminate form \( 0/0 \) when we plug in \( x = 0 \), so we apply L'Hopital’s rule to get

\[ \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} = \lim_{x \to 0} \frac{\sin x}{x \sin x + x \cos x}. \]

Plugging in \( x = 0 \) still gives us the indeterminate form \( 0/0 \), so we apply L'Hopital’s rule again to get

\[ \lim_{x \to 0} \frac{\sin x}{\sin x + x \cos x} = \lim_{x \to 0} \frac{\cos x}{\cos x + \cos x - x \sin x}. \]

Plugging in \( x = 0 \) here gives us \( \frac{1}{2} \). Therefore, our answer is \( \left( \frac{1}{2} \right)^2 = \frac{1}{4} \).

7. Compute

\[ \int_{-2}^{0} \frac{x^3 + 4x^2 + 7x - 20}{x^2 + 4x + 8} \, dx + \int_{0}^{2} \frac{2x^3 - 7x^2 + 9x - 10}{x^2 + 4} \, dx. \]

**Answer:** 12

**Solution:** The polynomials in the denominator are similar, so we rewrite them in the form

\[ \int_{-2}^{0} \frac{x^3 + 4x^2 + 7x - 20}{(x + 2)^2 + 4} \, dx + \int_{0}^{2} \frac{2x^3 - 7x^2 + 9x - 10}{x^2 + 4} \, dx. \]

This suggests we make the \( u \)-substitution \( u = x + 2 \) in the first integral, which gives us

\[ \int_{0}^{2} \frac{(u - 2)^3 + 4(u - 2)^2 + 7(u - 2) - 20}{u^2 + 4} \, du + \int_{0}^{2} \frac{2u^3 - 7u^2 + 9u - 10}{u^2 + 4} \, du. \]

which simplifies down to

\[ \int_{0}^{2} \frac{u^3 - 2u^2 + 3u - 26}{u^2 + 4} \, du + \int_{0}^{2} \frac{2u^3 - 7u^2 + 9u - 10}{u^2 + 4} \, du. \]

We can then combine the two integrals into a single integral and combine like terms to get

\[ \int_{0}^{2} \frac{3x^3 - 9x^2 + 12x - 36}{x^2 + 4} \, dx = \int_{0}^{2} (3x - 9) \, dx = \left( \frac{3}{2} x^2 - 9x \right) \bigg|_{0}^{2} = \frac{3}{2} (2^2) - 9 \cdot 2 = -12 \]

8. Compute

\[ \lim_{n \to \infty} n^2 \int_{0}^{1/n} x^{2018x+1} \, dx. \]

**Answer:** \( \frac{1}{2} \)
Solution: The key to evaluating this limit is to approximate the limit with a simpler integral. In particular, notice that $x^x \to 1$ as $x \to 0$. This suggests that we can approximate the integrand using $x^{2018x} = (x^x)^{2018} \approx 1$ to get

$$
\lim_{n \to \infty} n^2 \int_0^{1/n} x^{2018x+1} \, dx = \lim_{n \to \infty} n^2 \int_0^{1/n} x \, dx.
$$

The integral then evaluates to $\int_0^{1/n} x \, dx = \frac{1}{2n^2}$. Plugging this back into the limit gives us the answer $\frac{1}{2}$.

To prove this claim rigorously, let $\epsilon > 0$. Because $\lim_{x \to 0} x^{2018x} = 1$, there exists some $\delta > 0$ such that $|x^{2018x} - 1| < \epsilon$ for all $0 < x < \delta$. Now consider the integral

$$
n^2 \int_0^{1/n} x^{2018x+1} - x \, dx.
$$

Armed with our approximation, so long as $0 < \frac{1}{n} < \delta$, or $n > \frac{1}{\delta}$, we have

$$
\left| n^2 \int_0^{1/n} x^{2018x+1} - x \, dx \right| = \left| n^2 \int_0^{1/n} x (x^{2018x} - 1) \, dx \right|
\leq n^2 \int_0^{1/n} x |x^{2018x} - 1| \, dx
\leq n^2 \int_0^{1/n} \epsilon \cdot x \, dx
= \frac{\epsilon}{2}
$$

where we have used the facts that the function $x$ is non-negative on the interval $[0, 1/n]$.

We have already shown above that $n^2 \int_0^{1/n} x \, dx = \frac{1}{2}$, so for all $n > \frac{1}{\delta}$ we have

$$
\left| n^2 \int_0^{1/n} x^{2018x+1} \, dx - \frac{1}{2} \right| < \frac{\epsilon}{2}.
$$

This is precisely the definition of a limit, so we conclude that the limit does exist and approaches $\frac{1}{2}$ as claimed above.

9. Compute

$$
\int_0^{\pi} \frac{2x \sin x}{3 + \cos 2x} \, dx.
$$

Answer: $\frac{\pi^2}{4}$

Solution: Let

$$
I = \int_0^{\pi} \frac{2x \sin x}{3 + \cos 2x} \, dx.
$$

We first use the identity $\cos 2x = 2 \cos^2 x - 1$ to reduce the integral into the form

$$
\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx.
$$

Next, we consider the substitution $x \mapsto \pi - y$. The integral then becomes

$$
\int_0^{\pi} \frac{(\pi - y) \sin y}{1 + \cos^2 y} \, dy.
$$
Adding the two integrals together, we get
\[ 2I = \int_{0}^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} \, dx. \]
This gives us the simpler
\[ I = \frac{\pi}{2} \int_{0}^{\pi} \frac{\sin x}{1 + \cos^2 x} \, dx. \]
Now use the substitution \( u = \cos x \) to get
\[ -\frac{\pi}{2} \int_{1}^{-1} \frac{1}{1 + u^2} \, du. \]
This is the standard \( \tan^{-1} u \) integral, so we have
\[ I = \frac{\pi}{2} \left( \tan^{-1}(1) - \tan^{-1}(-1) \right) = \frac{\pi}{2} \left( \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right) = \frac{\pi^2}{4}. \]

10. **Fact:** The value \( \ln(2) \) is not the root of any polynomial with rational coefficients.

For any nonnegative integer \( n \), let \( p_n(x) \) be the unique polynomial with integer coefficients such that
\[ p_n(\ln(2)) = \int_{1}^{2} (\ln(x))^n \, dx. \]
Compute the value of the sum
\[ \sum_{n=0}^{\infty} \frac{1}{p_n(0)}. \]

**Answer:** \( \frac{1}{e} \)

**Solution:** We first attempt to compute the integral \( p_n(\ln(2)) = \int_{1}^{2} (\ln(x))^n \, dx. \) Using integration by parts with \( u = (\ln(x))^n \) and \( dv = 1 \), we get
\[ \int_{1}^{2} (\ln(x))^n \, dx = x(\ln(x))^n \bigg|_{1}^{2} - \int_{1}^{2} n(\ln(x))^{n-1} \, dx \]
\[ = 2(\ln(2))^n - n \int_{1}^{2} (\ln(x))^{n-1} \, dx \]
\[ = 2(\ln(2))^n - np_{n-1}(\ln(2)). \]
Because \( \ln(2) \) is not the root of any polynomial with rational coefficients, we can plug in \( x = \ln(2) \) to get the recurrence
\[ p_n(x) = 2x^n - np_{n-1}(x). \]
Recall that we are only concerned with the constant terms \( p_n(0) \), so plugging in \( x = 0 \) yields
\[ p_n(0) = -np_{n-1}(0). \]
Now since
\[ p_0(\ln(2)) = \int_{1}^{2} \, dx = 1, \]
we have \( p_0(x) = 1 \), and therefore \( p_0(0) = 1 \). From the recurrence, we deduce that
\[ p_n(0) = (-1)^n \cdot n!. \]
Our desired sum is thus
\[ \sum_{n=0}^{\infty} \frac{1}{p_n(0)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}, \]
from the Taylor series expansion of \( e^x \) at \( x = -1 \).