1. If $a, b, c$ are real numbers with $a - b = 4$, find the maximum value of $ac + bc - c^2 - ab$.

**Answer:** 4

**Solution:** We have

$$ac + bc - c^2 - ab = (a - c)(c - b)$$

$$= \left(\sqrt{(a - c)(c - b)}\right)^2$$

$$\leq \left(\frac{(a - c) + (c - b)}{2}\right)^2$$

$$= \left(\frac{a - b}{2}\right)^2$$

$$= \left(\frac{4}{2}\right)^2$$

$$= 4$$

The maximum is attained when $a = 4$, $b = 0$, and $c = 2$.

2. If $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$ and $\frac{1}{x+1} + \frac{1}{y+1} = \frac{3}{8}$, compute $\frac{1}{x-1} + \frac{1}{y-1}$.

**Answer:** $\frac{11}{14}$

**Solution:** With two equations and two unknowns, we can solve for the expressions $a = xy$ and $b = x + y$. The first equation can be written as

$$\frac{1}{x} + \frac{1}{y} = \frac{x + y}{xy} = \frac{b}{a} = \frac{1}{2}$$

which implies that $a = 2b$. Similarly, from the second equation we have

$$\frac{1}{x + 1} + \frac{1}{y + 1} = \frac{(x + y) + 2}{xy + (x + y) + 1} = \frac{b + 2}{a + b + 1} = \frac{3}{8}$$

which implies that $8(b + 2) = 3(a + b + 1)$, or $3a - 5b = 13$. Solving the system of equations gives us $a = 26$ and $b = 13$. Therefore, we have

$$\frac{1}{x - 1} + \frac{1}{y - 1} = \frac{(x + y) - 2}{xy - (x + y) + 1} = \frac{b - 2}{a - b + 1} = \frac{11}{14}$$

3. Let $F_n$ denote the $n$-th term of the Fibonacci sequence defined recursively as $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Compute the sum

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n}$$

**Answer:** 2
Let $S$ be the desired sum. Note that

$$S - \frac{S}{2} = \sum_{n=1}^{\infty} \frac{F_n}{2^n} - \sum_{n=1}^{\infty} \frac{F_n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{F_n}{2^n} - \sum_{n=2}^{\infty} \frac{F_{n-1}}{2^n}$$

$$= \frac{F_1}{2} + \frac{F_2}{4} + \sum_{n=3}^{\infty} \frac{F_n}{2^n} - \frac{F_1}{4} - \sum_{n=3}^{\infty} \frac{F_{n-1}}{2^n}$$

$$= \frac{1}{2} + \sum_{n=3}^{\infty} \frac{F_{n-2}}{2^n}$$

$$= \frac{1}{2} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{F_n}{2^n}$$

$$= \frac{1}{2} + \frac{S}{4}$$

Therefore, we have $\frac{S}{2} = \frac{1}{2} + \frac{S}{4}$, which solves to $S = 2$. 