1. At the grocery store, 3 avocados and 2 pineapples cost $8.80, while 5 avocados and 3 pineapples cost $14.00. How much do 1 avocado and 1 pineapple cost in dollars?

Answer: $3.60

Solution: Since 3 avocados and 2 pineapples cost $8.80, 6 avocados and 4 pineapples cost $17.60. Subtracting away the cost of 5 avocados and 3 pineapples, we have that 1 avocado and 1 pineapple cost $3.60.

2. Let $a$, $b$, $c$, $d$ be an increasing sequence of numbers such that $a$, $b$, $c$ forms a geometric sequence and $b$, $c$, $d$ forms an arithmetic sequence. Given that $a = 8$ and $d = 24$, what is $b$?

Answer: 12

Solution: Let $r$ be the common ratio so that $b = ar$ and $c = ar^2$. Then $d = 2c - b = 2ar^2 - ar$. Plugging in the values for $a$ and $d$, we get the quadratic $16r^2 - 8r - 24 = 0$. Factoring, we get $8(r + 1)(2r - 3) = 0$. Because the sequence is increasing, $r$ must be positive, so $r = \frac{3}{2}$. Therefore, the answer is $b = 8 \cdot \frac{3}{2} = 12$.

3. Given that the roots of the polynomial $x^3 - 7x^2 + 13x - 7 = 0$ are $r$, $s$, $t$, compute the value of $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$.

Answer: $\frac{13}{7}$

Solution 1: Vieta’s formulas give us the relations $r + s + t = 7$, $rs + st + rt = 13$ and $rst = 7$. If we write the fraction over a single denominator, we find

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{rs + st + rt}{rst} = \frac{13}{7}$$

Solution 2: Notice that we can factorize $x^3 - 7x^2 + 13x - 7$ into $(x - 1)(x^2 - 6x + 7)$. The roots are therefore $1$, $3 \pm \sqrt{2}$, allowing us to compute

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = \frac{1}{1} + \frac{1}{3 + \sqrt{2}} + \frac{1}{3 - \sqrt{2}} = 1 + \frac{3 - \sqrt{2} + 3 + \sqrt{2}}{9 - 2} = \frac{13}{7}$$

4. Find all possible pairs of integers $(m, n)$ which satisfy $m^2 + 2m - 35 = 2^n$.

Answer: $(9, 6), (-11, 6)$

Solution: Factoring, we have $(m + 7)(m - 5) = 2^n$. We observe that $m + 7$ and $m - 5$ are 12 apart, and also observe that the only powers of 2 which differ by 12 are 4 and 16, so $n = 2 + 4 = 6$. There are two cases:

(a) $m + 7 = 16$ and $m - 5 = 4$, which gives us $m = 9$.

(b) $m + 7 = -4$ and $m - 5 = -16$, which gives us $m = -11$.

The answer is therefore $(9, 6), (-11, 6)$.

5. Let $a_1, a_2, a_3, a_4, a_5, \ldots$ be a geometric progression with positive ratio such that $a_1 > 1$ and $(a_{1357})^3 = a_{34}$. Find the smallest integer $n$ such that $a_n < 1$.

Answer: 2019

Solution: Let $r$ be the ratio between the terms. The $n$-th term of the sequence can therefore be written as $a_n = a_1r^{n-1}$. This allows us to write

$$(a_{1357})^3 = a_{34} \Rightarrow (a_1r^{1356})^3 = a_1r^{33} \Rightarrow a_1^2r^{4035} = 1.$$
Since $a_1 > 1$, we must have $0 < r < 1$, so the geometric progression is decreasing. However, the above equation also gives
\[ a_1^2 r^{4035} = (a_1 r^{2017})(a_1 r^{2018}) = a_{2018}a_{2019} = 1. \]
Because the sequence is decreasing, we must have $a_{2018} > 1 > a_{2019}$, which gives us the answer 2019.

6. Let $a_k = \pm 1$ for all integers $1 \leq k \leq 2018$. The sum
\[ \sum_{1 \leq i < j \leq 2018} a_i a_j \]
can take on both positive and negative values. Find the smallest positive value of the sum.

**Answer:** 49

**Solution:** Observe that:
\[ \sum_{1 \leq i < j \leq 2018} a_i a_j = (a_1 + a_2 + ... + a_{2018})^2 - (a_1^2 + a_2^2 + ... + a_{2018}^2) \]
\[ = (a_1 + a_2 + ... + a_{2018})^2 - 2018 \]
Because $a_k = \pm 1$, we know that $a_1 + a_2 + ... + a_{2018}$ is an integer between $-2018$ and 2018 inclusive. Furthermore, there are an even number of terms, each of which is odd, so this sum is even.

Therefore, the minimum positive integer value of $(a_1 + a_2 + ... + a_{2018})^2 - 2018$ is 98, obtained when $a_1 = a_2 = ... = a_{46} = 1$ and the rest of the terms $a_{47}, a_{48}, ..., a_{2018}$ contain an equal number of 1’s and -1’s. Therefore, the least positive value is $\frac{98}{2} = 49$.

7. Let $x, y, z$ be non-negative real numbers satisfying $xyz = \frac{2}{3}$. Compute the minimum value of
\[ x^2 + 6xy + 18y^2 + 12yz + 4z^2. \]

**Answer:** 18

**Solution:** We first complete the square and rewrite our equation as
\[ (x + 3y)^2 + (3y + 2z)^2. \]
We then substitute $a = x$, $b = 3y$, and $c = 2z$ to minimize the equivalent sum
\[ (a + b)^2 + (b + c)^2 \]
under the condition $abc = 6xyz = 4$. Applying AM-GM gives us
\[ (a + b)^2 + (b + c)^2 \geq 2(a + b)(b + c). \]
We can apply AM-GM again to $a + b$ and $b + c$ individually via
\[ a + b = \frac{a}{2} + \frac{a}{2} + b \geq 3\sqrt[3]{\frac{a^2b}{4}} \]
\[ b + c = b + \frac{c}{2} + \frac{c}{2} \geq 3\sqrt[3]{\frac{bc^2}{4}} \]
to get
\[ (a + b)^2 + (b + c)^2 \geq 2(a + b)(b + c) \geq 18\sqrt[3]{\frac{(abc)^2}{16}} = 18 \]
Note that equality holds when $a = 2b = c$, so $a = c = 2$, and $b = 1$, or equivalently when $x = 2$, $y = \frac{1}{3}$, and $z = 1$. 

8. Define \( \{x\} = x - \lfloor x \rfloor \), where \( \lfloor x \rfloor \) denotes the largest integer not exceeding \( x \). If \( |x| \leq 8 \), find the number of real solutions to the equation
\[
\{x\} + \{x^2\} = 1.
\]

**Answer:** 113

**Solution:** Since \( \{x\} + \{x^2\} = 1 \), the value \( x \) must satisfy \( x + x^2 = n \) for some integer \( n \). The quadratic equation then gives us
\[
x = \frac{-1 \pm \sqrt{1 + 4n}}{2}.
\]
If we consider when \( 0 \leq x \leq 8 \), then we must have \( \frac{-1 + \sqrt{1 + 4n}}{2} \leq 8 \). Solving the inequality, we find that \( x \) satisfies the equation when \( 0 \leq n \leq 72 \), giving us 73 possibilities. Likewise, when \( -8 \leq x < 0 \), we must have \( \frac{-1 - \sqrt{1 + 4n}}{2} \geq -8 \), which has solutions when \( 0 \leq n \leq 56 \), for a total of 57 possibilities.
Since \( \{x\} + \{x^2\} < 2 \), we must also eliminate the cases when \( \{x\} + \{x^2\} = 0 \), which happens only when \( -8 \leq x \leq 8 \) is an integer, for a total of 17 possibilities.
Therefore, the total number of solutions is \( 73 + 57 - 17 = 113 \).

9. Let \((a, b, c, d, e)\) be an integer solution to the system of equations
\[
\begin{align*}
a + d &= 12 \\
b + ad + e &= 57 \\
c + bd + ae &= 134 \\
(cd + be) &= 156 \\
be &= 72
\end{align*}
\]
Find all possible values of \( b + d \).

**Answer:** 18, 21, 25

**Solution:** After much careful consideration, we notice that the first, second, and third equations contain \( a, b, c \), the second, third, and fourth equations contain \( ad, bd, cd \), and the third, fourth, and fifth equations contain \( ae, be, ce \). We also have \( d \) and \( e \) terms. This suggests that the system of equations was constructed somehow using the product
\[
(1 + a + b + c)(1 + d + e).
\]
This looks suspiciously like the factorization of the polynomial
\[
(x^3 + ax^2 + bx + c)(x^2 + dx + e).
\]
In fact, if we expand this polynomial, we get the (semi-magical)
\[
x^5 + (a + d)x^4 + (b + ad + e)x^3 + (c + bd + ae)x^2 + (cd + be)x + ce.
\]
Plugging our values in from our system of equations, we realize that the desired solutions are the different factorizations of the polynomial
\[
x^5 + 12x^4 + 57x^3 + 134x^2 + 156x + 72 = (x + 2)^3(x + 3)^2
\]
into the product of a cubic and a quadratic. We can do this in three ways, leading to the solutions

\[
[(x + 2)^3][(x + 3)^2] = (x^3 + 6x^2 + 12x + 8)(x^2 + 6x + 9) \implies (6, 12, 8, 6, 9)
\]

\[
[(x + 2)^2(x + 3)][(x + 2)(x + 3)] = (x^3 + 7x^2 + 16x + 12)(x^2 + 5x + 6) \implies (7, 16, 12, 5, 6)
\]

\[
[(x + 2)(x + 3)^2][(x + 2)^2] = (x^3 + 8x^2 + 21x + 18)(x^2 + 4x + 4) \implies (8, 21, 18, 4, 4)
\]

Therefore, the three possible values of \( b + d \) are \([18, 21, 25]\).

10. Let \( a_1, \ldots, a_{2018} \) be the roots of the polynomial

\[
x^{2018} + x^{2017} + \cdots + x^2 + x - 1345 = 0.
\]

Compute

\[
\sum_{n=1}^{2018} \frac{1}{1 - a_n}.
\]

**Answer:** 3027

**Solution:** We begin by defining \( b_n = \frac{1}{1 - a_n} \). Rearranging gives us \( a_n = b_n - 1 \). Since we know \(-1346 + \sum_{k=0}^{2018} a_n^k = 0 \) for all \( 1 \leq n \leq 2018 \), we can substitute \( b_n \) in to get a new polynomial

\[
\sum_{k=0}^{2018} \left( \frac{b_n - 1}{b_n} \right)^k - 1346 = 0 \implies \sum_{k=0}^{2018} (b_n)^{2018-k}(b_n - 1)^k - 1346b_n^{2018} = 0
\]

where we have multiplied both sides by \( b_n^{2018} \) which is nonzero because \( a_n \neq 1 \). This is true for all \( 1 \leq n \leq 2018 \), so \( b_n \) are in fact the roots of the polynomial

\[
\sum_{k=0}^{2018} x^{2018-k}(x - 1)^k - 1346x^{2018} = 0.
\]

By Vieta’s it is enough to calculate the coefficients of \( x^{2018} \) and \( x^{2017} \) in the polynomial to compute the sum of the roots. We see that the coefficient of \( x^{2018} \) is \( 2019 - 1346 = 673 \) and the coefficient of \( x^{2017} \) is \(-1 - 2 - \cdots - 2018 = -\frac{2019 \cdot 2018}{2} \), which gives us the answer \( \frac{2018 \cdot 2019}{2 \cdot 673} = 3027 \).