1. Alice and Bob are painting a house. If Alice and Bob do not take any breaks, they will finish painting the house in 20 hours. If, however, Bob stops painting once the house is half-finished, then the house takes 30 hours to finish. Given that Alice and Bob paint at a constant rate, compute how many hours it will take for Bob to paint the entire house if he does it by himself.

   Answer: 40

   Solution: In 10 hours, Alice and Bob paint half the house. Therefore, Alice can paint half the house in 20 hours. This means Alice painted a quarter of the house in 10 hours, which means Bob paints a quarter of the house in 10 hours, so Bob takes $\boxed{40}$ hours to paint the entire house.

2. Compute $9^6 + 6 \cdot 9^5 + 15 \cdot 9^4 + 20 \cdot 9^3 + 15 \cdot 9^2 + 6 \cdot 9$.

   Answer: 999999

   Solution: From the Binomial Theorem, this is just $(9+1)^6 - 1 = \boxed{999999}$.

3. Let x_1 and x_2 be the roots of $x^2 - x - 2014$, with $x_1 < x_2$. Let x_3 and x_4 be the roots of $x^2 - 2x - 2014$, with $x_3 < x_4$. Compute $(x_4 - x_2) + (x_3 - x_1)$.

   Answer: 1

   Solution: Note that $x_3 + x_4 = 2$ and $x_1 + x_2 = 1$, giving an answer of $\boxed{1}$.

4. For any 4-tuple (a_1, a_2, a_3, a_4) where each entry is either 0 or 1, call it *quadratically satisfiable* if there exist real numbers $x_1, \ldots, x_4$ such that $x_1 x_4^2 + x_2 x_4 + x_3 = 0$ and for each $i = 1, \ldots, 4$, x_i is positive if $a_i = 1$ and negative if $a_i = 0$. Find the number of *quadratically satisfiable* 4-tuples.

   Answer: 12

   Solution: First, we may assume $a_1 = 1$ without loss of generality and multiply our answer by 2 at the end, since $ax^2 + bx + c = 0 \Leftrightarrow -ax^2 - bx - c = 0$. We can furthermore assume $x_1 = 1$, since we can always divide the whole equation by x_1 (since $x_1 > 0$).

   Hence, we now consider equations of the form $x_4^2 + bx_4 + c = 0$ in which b and c are constrained to be either positive or negative. This yields four cases:

   Case 1: If b and c are both positive, the two roots have positive product but negative sum, so they must both be negative i.e. $x_4 < 0$. Furthermore, $x_4 < 0$ is possible, e.g. $x_4^2 + 2x_4 + 1 = 0 \implies x_4 = -1$.

   Case 2: If b is positive and c is negative, x_4 may be positive or negative e.g. $x_4^2 + x_4 - 2 \implies x_4 \in \{-2, 1\}$.

   Case 3: If b is negative and c is positive, the two roots have positive product and positive sum, so they must both be positive i.e. $x_4 > 0$. Furthermore, $x_4 > 0$ is possible e.g. $x_4^2 - 2x_4 + 1 \implies x_4 = 1$.

   Case 4: If b and c are both negative, x_4 may be positive or negative e.g. $x_4^2 - x_4 - 2 \implies x_4 \in \{-1, 2\}$.

   Putting these cases together, we conclude that the answer is $\boxed{12}$.

5. a and b are nonnegative real numbers such that $\sin(ax + b) = \sin(29x)$ for all integers x. Find the smallest possible value of a.

   Answer: $10\pi - 29$.

Solution: First, since $\sin(b) = \sin(0) = 0$, we have $b = n\pi$ for some integer n. Since sin has period 2π, we need only consider the cases when $b = 0$ and $b = \pi$.

Now let $b \in \{0, \pi\}$ and a be any real number. If for all integers x, $\sin(ax + b) = \sin(29x)$, then for any integer n,

$$\sin((a + 2\pi n)x + b) = \sin(ax + b + 2\pi nx) = \sin(ax + b) = \sin(29x)$$

for all integers x as well. Conversely, assume for some a and c that for all integers x, $\sin(ax+b) = \sin(cx + b) = \sin(29x)$. Then, for all integers x,

$$\begin{aligned}\sin(ax) &= \frac{\sin(ax)\cos(b) + \cos(ax)\sin(b)}{\cos(b)} \\ &= \frac{\sin(ax + b)}{\cos(b)} \\ &= \frac{\sin(cx + b)}{\cos(b)} \\ &= \frac{\sin(cx)\cos(b) + \cos(cx)\sin(b)}{\cos(b)} = \sin(cx),\end{aligned}$$

since $\sin(0) = \sin(\pi) = 0$ and $\cos(0), \cos(\pi) \neq 0$. But then, $\sin(a) = \sin(c)$ and $2\sin(a)\cos(a) = \sin(2a) = \sin(2c) = 2\sin(c)\cos(c)$ implies $\cos(a) = \cos(c)$ since $\sin(a) = \sin(c) = \frac{\sin(29)}{\cos(b)} \neq 0$. Hence, a and c are the same angle, modulo integer multiples of 2π.

Now, we consider the two cases concretely. If $b = 0$, one valid assignment of a is $a = 29$, so all possible ones are $a = 29 + 2\pi n$ for integers n. The smallest positive number we can make this is $29 - 8\pi$, since $10\pi \approx 31.4 > 29$.

Meanwhile, if $b = \pi$, one valid assignment of a is $a = -29$, since $\sin(-29x+\pi) = \sin(-29x)\cos(\pi) + \cos(-29x)\sin(\pi) = -\sin(-29x) = \sin(29x)$. So, all possible ones are $a = -29 + 2\pi n$ for integers n. The smallest positive number we can make this is $\boxed{10\pi - 29}$. We can easily see that $29 \in (9\pi, 10\pi)$, so $10\pi - 29 < \pi < 29 - 8\pi$.

6. Find the minimum value of
$$\frac{1}{x-y} + \frac{1}{y-z} + \frac{1}{x-z}$$
for reals $x > y > z$ given $(x - y)(y - z)(x - z) = 17$.

Answer: $\mathbf{\frac{5}{\sqrt[3]{68}}}$

Solution: Let $x - y = a$ and $y - z = b$. Therefore, $x - z = a + b$.

Note that $\frac{1}{a} + \frac{1}{b} + \frac{1}{a+b} = 4 \cdot \frac{(a+b)}{4ab} + \frac{1}{a+b}$. Applying AM-GM here on these five fractions, this quantity is greater than or equal to $5\sqrt[5]{\frac{(a+b)^3}{256a^4b^4}}$. Note that because $ab(a + b) = 17$, this is equivalent to minimizing $\sqrt[5]{\frac{(a+b)^7}{256 \cdot 17^4}}$, which means that we want to minimize $a + b$, which implies that we want $a = b$ because that is when the inequality is tight.

We therefore have that $17 = ab(a + b) \leq \frac{(a+b)^3}{4}$ because $\frac{(a+b)^2}{4} \geq ab$ by AM-GM. Note that we have equality if and only if $a = b$, which implies that $a + b = \sqrt[3]{68}$, so $a = b = \frac{\sqrt[3]{68}}{2}$. Plugging this in gives us $\boxed{\frac{5}{\sqrt[3]{68}}}$.

7. Compute the smallest value p such that, for all $q > p$, the polynomial $x^3 + x^2 + qx + 9$ has exactly one real root.

Answer: $-\frac{\mathbf{39}}{\mathbf{4}}$

Solution:

Let $f(x) = x^3 + x^2 + px + 9$. Then $f(x)$ must have a negative root a and a double root b. By viete's, we have the following equations:

$$ab^2 = -9$$

$$a + 2b = -1$$

This gives the cubic $(2b+1)b^2 = 9 \Rightarrow 2b^3 + b^2 - 9 = 0$. This equation yields $b = \frac{3}{2}$ as the only real solution, so $a = -4$ and $p = \boxed{-\frac{39}{4}}$.

8. $P(x)$ and $Q(x)$ are two polynomials such that

$$P(P(x)) = P(x)^{16} + x^{48} + Q(x).$$

Find the smallest possible degree of Q.

Answer: 35

Solution: Note: we use the notation $O(x^n)$ to denote an arbitrary polynomial whose degree is at most n.

We first try to find a Q with degree < 48. It turns out this is feasible. Let d be the degree of P. $P(P(x))$ has degree d^2, and $P(x)^{16} + x^{48} + Q(x)$ has degree $\max(16d, 48)$. Since 48 is not a perfect square, the degree must be $16d$, which implies $d = 16$.

Now let $R(x) = P(x) - x^{16}$, so

$$R(P(x)) = x^{48} + Q(x).$$

Since R applied to a degree-16 polynomial yields a degree-48 polynomial, the degree of R must be 3. So, we have $P(x) = x^{16} + ax^3 + O(x^2)$ for some $a \neq 0$; we can also show from here that in fact $a = 1$. Therefore,

$$P(P(x)) = P(x)^{16} + P(x)^3 + O(P(x)^2) = P(x)^{16} + x^{48} + 3x^{35} + O(x^{34}).$$

Hence, if the degree of Q is < 48, it must be exactly $\boxed{35}$.

9. Let b_n be defined by the formula

$$b_n = \sqrt[3]{-1 + a_1\sqrt[3]{-1 + a_2\sqrt[3]{-1 + \ldots a_{n-1}\sqrt[3]{-1 + a_n}}}}$$

where $a_n = n^2 + 3n + 3$. Find the smallest real number L such that $b_n < L$ for all n.

Answer: 3

Solution: One way of solving this problem is by noticing the identity

$$n+2=\sqrt[3]{(n+2)^3+1-1}=\sqrt[3]{-1+(n+2)^3+1}=\sqrt[3]{-1+((n+2)^2-(n+2)+1)(n+3)}=$$
$$=\sqrt[3]{-1+(n^2+3n+3)(n+3)}=\sqrt[3]{-1+a_n(n+3)}$$

It is quite easy to see that $n+k+2=\sqrt[3]{-1+a_{n+k}(n+k+3)}$, so the formula may be applied recursively to obtain the result

$$3=\sqrt[3]{-1+a_1\sqrt[3]{-1+a_2\sqrt[3]{-1+\ldots+a_{k-1}\sqrt[3]{-1+a_k(k+3)}}}}$$

for arbitrary $k\geq 1$. Then for all $n\geq 1$,

$$\sqrt[3]{-1+a_1\sqrt[3]{-1+\ldots\sqrt[3]{-1+a_n}}}<\sqrt[3]{-1+a_1\sqrt[3]{-1+\ldots+\sqrt[3]{-1+a_n(n+3)}}}=3$$

This gives a pretty good candidate for L.

Next, it is pretty clear that b_n is an increasing (just by checking what happens in the innermost radicals), and the upper bound of 3 implies that b_n approaches some number ≤ 3 for large n-essentially, this is intuitive justification for the existence of L. This also motivates checking if $L=3$ or not by the following way:

Define $b_n(k)$ as the same formula for b_n with n roots, but instead of starting at a_1, it starts at n_k. Using computations very similar to those above, we may determine that, more generally,

$$b_n(k)<k+2$$

and that $b_n(k)$ increases as n increases for any fixed k. Next, define

$$c_n(k)=k+2-b_n(k).$$

If $c_n(k)$ gets arbitrarily close to 0, then L cannot be less than 3, which would prove that $L=3$. We compute

$$\begin{aligned}
c_n(k)=k+2-b_n(k)=k+2-\sqrt[3]{-1+a_kb_{n-1}(k+1)}&=\frac{(k+2)^3+1-a_kb_{n-1}(k+1)}{(k+2)^2+(k+2)b_n(k)+b_n(k)^2}\\
&=\frac{(k+3)((k+2)^2-(k+2)+1)-a_kb_{n-1}(k+1)}{(k+2)^2+(k+2)b_n(k)+b_n(k)^2}=\frac{a_k((k+3)-b_{n-1}(k+1))}{(k+2)^2+(k+2)b_n(k)+b_n(k)^2}\\
&=\frac{a_kc_{n-1}(k+1)}{(k+2)^2+(k+2)b_n(k)+b_n(k)^2}<\frac{a_kc_{n-1}(k+1)}{(k+2)^2+(k+2)+1}\\
&=\frac{a_kc_{n-1}(k+1)}{k^2+5k+7}=\frac{a_k}{a_{k+1}}c_{n-1}(k+1).
\end{aligned}$$

I used the fact that $b_n(k) > 1$ which is true because $a_k > 2$ for $k \geq 1$, and by replacing all the a_i with 2 in the expression for $b_n(k)$ you get simply 1. Applying this inequality repeatedly, we get

$$\begin{aligned} c_n(k) &< \frac{a_k}{a_{k+1}}\frac{a_{k+1}}{a_{k+2}}\cdots\frac{a_{n+k-2}}{a_{n+k-1}}c_1(n+k-1) = \frac{a_k}{a_{n+k-1}}c_1(n+k-1) \\ &= \frac{a_k}{a_{n+k-1}}(n+k+1-b_1(n+k-1)) = \frac{(k^2+3k+3)(n+k+1-\sqrt[3]{-1+a_{n+k-1}})}{(n+k-1)^2+3(n+k-1)+3} \\ &= \frac{1}{n}\frac{(k^2+3k+3)(1+k/n+1/n-\sqrt[3]{-1/n^3+a_{n+k-1}/n^3})}{(1+k/n-1/n)^2+3(1/n+k/n^2-1/n^2)+3/n^2}. \end{aligned}$$

From this expression it is clear that, for any fixed k, for very large n $c_n(k)$ will get arbitrarily close to 0. The fraction multiplied by the $\frac{1}{n}$ has denominator approaching 1 and numerator approaching k^2+3k+3, as n becomes very large, because $k/n \to 0$, $1/n \to 0$ and $a_{n+k-1}/n^3 = ((n+k-1)^2+(n+k-1)+1)/n^3 \to 0$. So for large n, we may approximate the expression with

$$\frac{1}{n}\cdot(k^2+3k+3) \to 0.$$

Thus, $b_n(k)$ can get arbitrary close to $k+2$ but never reach it, and the case $k=1$ gives us the result that $L = \boxed{3}$.

10. Let $x_0 = 1, x_1 = 0$, and $x_i = -3x_{i-1} + x_{i-2}$ for $i \geq 2$. Let $y_0 = 0, y_1 = 1$, and $y_i = -3y_{i-1} + y_{i-2}$ for $i \geq 2$. Compute

$$\sum_{i=0}^{2013}\frac{(x_i y_{2014} - y_i x_{2014})^2}{y_{2014}^2}.$$

You may give your answer in terms of at most ten values of the x_i and/or y_i (but must otherwise simplify completely).

Answer: $\frac{3y_{2014}-x_{2014}}{3y_{2014}} = -\frac{y_{2015}}{3y_{2014}}$

Solution 1: Let $a = -x_{2014}/y_{2014}$.

We first show that $x_i + ay_i > 0$ for all i. Solving the linear recurrences gives

$$x_i = \frac{(-1)^i(-3+\sqrt{13})}{2\sqrt{13}}\left(\frac{3+\sqrt{13}}{2}\right)^i + \frac{3+\sqrt{13}}{2\sqrt{13}}\left(\frac{-3+\sqrt{13}}{2}\right)^i,$$

$$y_i = -\frac{(-1)^i}{\sqrt{13}}\left(\frac{3+\sqrt{13}}{2}\right)^i + \frac{1}{\sqrt{13}}\left(\frac{-3+\sqrt{13}}{2}\right)^i.$$

By cross-multiplying and cancelling terms, we conclude that

$$\frac{x_i}{y_i} - \frac{\frac{-3+\sqrt{13}}{2\sqrt{13}}}{-\frac{1}{\sqrt{13}}} = \frac{(-3+\sqrt{13})^i}{\sqrt{13}\left(-\frac{(-1)^i}{\sqrt{13}}(3+\sqrt{13})^i + \frac{1}{\sqrt{13}}(-3+\sqrt{13}))^i\right)}.$$

Since $-3+\sqrt{13} < -3+\sqrt{16} = 1$ and the denominator is $2^i y_i\sqrt{13}$, this number decreases monotonically in magnitude as i increases and alternates in sign. That is, as i increases, x_i/y_i gets monotonically closer to $(3-\sqrt{13})/2$ while alternating between being slightly above and slightly below. This means that $-x_{2i+1}/y_{2i+1} < a < -x_{2i}/y_{2i}$ for all $i \leq 1006$, as desired.

Hence consider the sequence of rectangles $R_0, R_1, \ldots, R_{2013}$, where R_{2i} has height $x_{2i} + ay_{2i}$ and width $3(x_{2i} + ay_{2i})$ and R_{2i+1} has height $3(x_{2i+1} + ay_{2i+1})$ and width $x_{2i+1} + ay_{2i+1}$. Draw R_{2i+1} adjacent to R_{2i} to the right with bottom edges aligned, and R_{2i+2} adjacent to R_{2i+1} above with left edges aligned. Then the entire drawing exactly forms a rectangle of height 1 and width $3 + a$, hence area $3 + a$. On the other hand the area of the rectangle is clearly 3 times the area of the desired sum. Therefore the sum has value $\frac{3+a}{3} = \boxed{\dfrac{3y_{2014} - x_{2014}}{3y_{2014}}}$.

Solution 2: Solving the linear recurrences, plugging in, and expanding results in a sum of a few geometric series. It should be possible to bash through this to get the same answer.