1. A circle of radius 2 is inscribed in equilateral triangle $ABC$. The altitude from $A$ to $BC$ intersects the circle at a point $D$ not on $BC$. $BD$ intersects the circle at a point $E$ distinct from $D$. Find the length of $BE$.

**Answer:** $\frac{6}{\sqrt{7}}$

**Solution:** We can find the side length of the triangle as follows: Since the triangle is equilateral, the segment from the center of the circle to a vertex (say $A$) bisects the angle. Also, the segment from the center to an adjacent point of tangency (say the perpendicular to $AC$) creates a right angle. So we get a 30-60-90 triangle, which tells us that the side length of the triangle is $4\sqrt{3}$.

Next, we call the point of intersection of $AD$ and $BC$ point $F$. Consider triangle $DBF$. Using the Pythagorean theorem, $BD = 2\sqrt{7}$. By power of a point, $BE \cdot BD = BF^2$. So $BE = \frac{BF^2}{BD} = \frac{12}{2\sqrt{7}} = \frac{6}{\sqrt{7}}$.

2. Points $A$, $B$, and $C$ lie on a circle of radius 5 such that $AB = 6$ and $AC = 8$. Find the smaller of the two possible values of $BC$.

**Answer:** $\frac{14}{5}$

**Solution:** Fix segment $AB$, and let $C$ and $D$ be the two points on the circle 8 units from $A$, where $C$ is closer to $B$ than $D$. Observe that $BD$ is a diameter (and hence $BD = 10$) because a 6-8-10 inscribed right triangle must be possible.

Next, we see that $CD$ can be calculated by drawing the diameter that goes through $A$, intersecting the opposite side of the circle at point $E$. Note that $CD \perp AE$. Moreover, $ACE$ and $ADE$ are right, since they are inscribed in semicircles, and $EC = ED = 6$ by the Pythagorean Theorem. Computing the area of quadrilateral $ACED$ two different ways, we get

$$\frac{1}{2} \cdot AE \cdot CD = 5 \cdot CD = \frac{1}{2} \cdot AC \cdot CE + \frac{1}{2} \cdot AD \cdot DE = 48 \implies CD = \frac{48}{5}.$$

Finally, since $BCD$ is a right triangle with $CD = \left(\frac{2}{5}\right)24$ and $BD = \left(\frac{2}{5}\right)25$, we conclude that $BC = \left(\frac{2}{5}\right)7 = \frac{14}{5}$.

3. In quadrilateral $ABCD$, diagonals $AC$ and $BD$ intersect at $E$. If $AB = BE = 5$, $EC = CD = 7$, and $BC = 11$, compute $AE$.

**Answer:** $\frac{47}{7}$

**Solution 1:** First, notice that length $AE$ is completely determined by the fact that $AB = BE$ and by the lengths of $AB$, $BC$ and $EC$. Thus, we only consider the triangle $ABC$. First, drop altitude $BH$ and note that since $ABE$ is isosceles, $EH = \frac{1}{2}AE$. Now, using Pythagoras twice, we have

$$BH^2 = 5^2 - EH^2 \quad \text{and} \quad BH^2 = 11^2 - (7 + EH)^2.$$ 

Setting these two equations to be equal, we can thus solve the equation $25 = 72 - 14EH$. Therefore, $AE = 2EH = \frac{47}{7}$.
Solution 2: Since $\angle AEB \cong \angle DEC$, we have $\triangle AEB \sim \triangle DEC$ by SAS. Hence, $\frac{AE}{DE} = \frac{BE}{CE} \implies \frac{AE}{BE} = \frac{DE}{CE}$. Additionally, $\angle BEC \cong \angle AED$, so $\triangle BEC \sim \triangle AED$ by SAS again.

Now, we do some angle-chasing. Since $\angle BAE \cong \angle CDE$ and $\angle EAD \cong \angle EBC$, $\angle ABC$ and $\angle ADC$ are supplementary. Hence, $ABCD$ is cyclic.

Let $AE = 5x$, so $DE = 7x$ since $\frac{AE}{DE} = \frac{BE}{CE} = \frac{5}{7}$. Also, note that $AD = x \cdot BC = 11x$ because $\frac{AD}{BC} = \frac{AE}{BE} = \frac{5x}{7} = x$.

Now, Ptolemy’s Theorem gives us

\[5 \cdot 7 + 11 \cdot 11x = (5x + 7)(7x + 5)\]
\[\implies 121x + 35 = 35x^2 + 74x + 35\]
\[\implies 121x = 35x^2 + 74x\]
\[\implies 47x = 35x^2\]
\[\implies x = \frac{47}{35}\]

because $x > 0$.

Hence, report $5x = \frac{47}{7}$. 