1. In triangle $ABC$, $AC = 7$. $D$ lies on $AB$ such that $AD = BD = CD = 5$. Find $BC$.

**Answer:** $\sqrt{51}$

**Solution:** Let $m\angle A = x$ and $m\angle B = y$. Note that we have two pairs of isosceles triangles, so $m\angle A = m\angle ACD$ and $m\angle B = m\angle BCD$. Since $m\angle ACD + m\angle BCD = m\angle ACB$, we have

$$180^\circ = m\angle A + m\angle B + m\angle ACB = 2x + 2y \implies m\angle ACB = x + y = 90^\circ.$$

Since $\angle ACB$ is right, we can use the Pythagorean Theorem to compute $BC$ as

$$\sqrt{10^2 - 7^2} = \sqrt{51}.$$

For a shortcut, note that $D$ is the circumcenter of $ABC$ and lies on the triangle itself, so it must lie opposite a right angle.

2. What is the perimeter of a rectangle of area 32 inscribed in a circle of radius 4?

**Answer:** $16\sqrt{2}$

**Solution:** It turns out the rectangle is actually a square with side length $4\sqrt{2}$, and hence has perimeter $16\sqrt{2}$.

3. Robin has obtained a circular pizza with radius 2. However, being rebellious, instead of slicing the pizza radially, he decides to slice the pizza into 4 strips of equal width both vertically and horizontally. What is the area of the smallest piece of pizza?

**Answer:** $\frac{\pi}{3} + 1 - \sqrt{3}$

**Solution 1:** Let $O$ be the center of the circle, and let $A$ and $B$ lie on the circle such that $m\angle AOB = 90^\circ$. Call $M$ the midpoint of $AO$ and $N$ the midpoint of $BO$. Let $C$ lie on minor arc $AB$ such that $CM \perp OA$, and let $D$ lie on minor arc $AB$ such that $DN \perp OB$. Finally, let $CM$ and $DN$ intersect at $E$. Now, the problem is to find the area of the region bounded by $AM$, $ME$, $ED$, and arc $DA$.

Notice that $ON = 1$ and $OD = 2$, so $OND$ is a 30-60-90 right triangle. Since $DN$ and $AO$ are parallel, $m\angle NDO = m\angle AOD = 30^\circ$. We now see that the area of the region bounded by $AM$, $ME$, $ED$, and arc $DA$ can be expressed as the sum of the areas of triangle $OND$ and sector $AOD$ minus the area of square $MONE$, which evaluates to

$$\frac{1}{2} \cdot 1 \cdot \sqrt{3} + \frac{\pi \cdot 2^2}{12} - 1 = \frac{\sqrt{3}}{2} + \frac{\pi}{3} - 1.$$

Finally, let $x$ denote the desired area. Then, the area of sector $AOB$ is

$$1 + 2 \left( \frac{\sqrt{3}}{2} + \frac{\pi}{3} - 1 \right) + x = \frac{\pi \cdot 2^2}{4} \implies x = \frac{\pi}{3} + 1 - \sqrt{3}.$$

**Solution 2:** When the pizza is sliced 4 times in both directions, the result is 4 unit squares, 8 congruent approximate quadrilaterals (one edge is curved), and 4 congruent approximate triangles (again, one edge is curved). Call the area of an approximate quadrilateral $x$ and an approximate triangle $y$. Since all these pieces form a circle of radius 2, we get

$$8x + 4y = 4\pi - 4.$$
Now, consider the long horizontal slice at the bottom of the pizza, consisting of 2 approximate quadrilaterals and 2 approximate triangles. Define the endpoints of the slice to be $A$ and $B$. Define the center of the pizza to be $C$. Consider the sector of the pizza cut out by $AC$ and $BC$. This is one third of the pizza, as $\angle ACB = 120^\circ$, and $\angle ABC = \angle BAC = 30^\circ$. Therefore, the area of the sector is $4\pi/3$ and the area of triangle $ABC$ is $\sqrt{3}$. Hence, we get

$$2x + 2y = \frac{4\pi}{3} - \sqrt{3}.$$ 

Therefore, we have the system

$$2x + y = \pi - 1$$
$$2x + 2y = \frac{4\pi}{3} - \sqrt{3}.$$ 

Solving this system gives

$$x = \frac{\pi}{3} - 1 + \frac{\sqrt{3}}{2}$$
$$y = \frac{\pi}{3} + 1 - \sqrt{3}.$$ 

Therefore, the smallest piece of pizza has area

$$\frac{\pi}{3} + 1 - \sqrt{3}.$$ 

4. $ABCD$ is a regular tetrahedron with side length 1. Find the area of the cross section of $ABCD$ cut by the plane that passes through the midpoints of $AB$, $AC$, and $CD$.

**Answer:** $\frac{1}{4}$

**Solution:** First, note that the plane also passes through the midpoint of $BD$ by symmetry, e.g. across the plane containing $AD$ perpendicular to $BC$. Let $M$, $N$, $O$, and $P$ denote the midpoints of $BA$, $AC$, $CD$, and $DB$, respectively. $MN = NO = OP = PM = \frac{1}{2}$ because they are all midlines of faces of the tetrahedron. Hence, the cross section is a rhombus. Furthermore, $MO \cong NP$ because both equal the distance between midpoints of opposite sides (alternatively, this congruence can be demonstrated by rotating $ABCD$ such that $N$ and $P$ coincide with the previous locations of $M$ and $O$). Hence, $MNOP$ is a square, and its area is $(\frac{1}{2})^2 = \frac{1}{4}$.

5. In square $ABCD$ with side length 2, let $P$ and $Q$ both be on side $AB$ such that $AP = BQ = \frac{1}{2}$. Let $E$ be a point on the edge of the square that maximizes the angle $PEQ$. Find the area of triangle $PEQ$.

**Answer:** $\frac{\sqrt{3}}{4}$

**Solution:** For any choice of $E$, we can draw the circumcircle of $PEQ$. Angle $PEQ$ is inscribed inside the minor arc of chord $PQ$, which is of constant length (it must always be the minor arc because $PEQ$ is clearly always acute). Therefore, maximizing $m\angle PEQ$ is equivalent to maximizing the measure of minor arc $PQ$, which in turn is equivalent to minimizing the radius of the circle.
Hence, we wish to find the smallest circle that intersects $ABCD$ at $P$, $Q$, and at least one other point. A circle of radius 1 can be tangent to sides $BC$ and $AD$, while a circle with a smaller radius clearly cannot touch any of the sides of the square. Hence, it is this circle we desire. Let this circle be centered at $O$. $OPQ$ is equilateral, so the height from $O$ to $PQ$ has length $\frac{\sqrt{3}}{2}$. This is also the height from the points of tangency on $AD$ or $BC$ to $PQ$. $E$ may be either one of these points, resulting in $PEQ$ having area $\sqrt{3} \cdot \frac{3}{4}$.

6. $ABCD$ is a rectangle with $AB = CD = 2$. A circle centered at $O$ is tangent to $BC$, $CD$, and $AD$ (and hence has radius 1). Another circle, centered at $P$, is tangent to circle $O$ at point $T$ and is also tangent to $AB$ and $BC$. If line $AT$ is tangent to both circles at $T$, find the radius of circle $P$.

**Answer:** $\frac{3 - \sqrt{5}}{2}$

**Solution:** Let the radius of circle $P$ be $r$. Draw $OP$, noting that it is perpendicular to $AT$ at $T$. Let $Q$ be the point of tangency between circle $O$ and $AD$. If we drop a perpendicular from $P$ to meet $OQ$ (extended) at $R$, then we know that $OR = 1 - r$ and $OP = 1 + r$, so by the Pythagorean theorem, $PR = 2\sqrt{r}$. Thus, $AQ = 2\sqrt{r} + r$.

Let $AB$ be tangent to $P$ at $U$. By the Two-Tangent Theorem, $AQ \cong AT \cong AU$. Since $UB = r$,

$$\sqrt{2} + r = 2 \implies r = \frac{3 - \sqrt{5}}{2}.$$

7. $ABCD$ is a square such that $AB$ lies on the line $y = x + 4$ and points $C$ and $D$ lie on the graph of parabola $y^2 = x$. Compute the sum of all possible areas of $ABCD$.

**Answer:** 68

**Solution 1:** First, shift the coordinate system so that the line goes through the origin and the parabola is now at $x = y^2 + 4$.

Let $CD$ lie on the line $y = x + b$. The distance between lines $AB$ and $CD$ is therefore $\frac{|b|}{\sqrt{2}}$, which can be proven by drawing 45-45-90 triangles. This distance is precisely $AD = BC$, so $CD$ must also have this length. Hence, the $y$-coordinates of $C$ and $D$ must have difference $\frac{|b|}{2}$, again by 45-45-90 triangles.

Substituting $x = y - b$ to $x = y^2 + 4$ yields $y^2 - y + (b + 4) = 0$. The difference between two solutions is $\sqrt{1 - 4(b + 4)} = \frac{b}{\sqrt{2}}$, which simplifies to $b^2 + 16b + 60 = 0$. The area of $ABCD$ is $\frac{1}{2}b^2$, so we want $\frac{1}{2}$ times the square of the possible values of $b$ as our answer. We can compute this as $\frac{16^2 - 2 \cdot 60}{2} = 68$.

**Solution 2:** Let $C = (y_1^2, y_1)$ and $D = (y_2^2, y_2)$, and assume without loss of generality that the points are positioned such that $y_1 < y_2$. Viewing this in the complex plane, we have $B - C = (D - C)i$, so $B = (y_1^2 + y_1 - y_2, y_2^2 - y_1^2 + y_1)$. Plugging this into $y = x + 4$ gives us $y_2^2 - 2y_1^2 + y_2 - 4 = 0$. Since $AB \parallel DC$, the slope of $DC$ is 1, so $\frac{y_2 - y_1}{y_1^2 - y_2^2} = 1 \implies y_1 + y_2 = 1$.

Solving this system of equations gives us two pairs of solutions for $(y_1, y_2)$, namely $(-1, 2)$ and $(-2, 3)$. These give $\sqrt{18}$ and $\sqrt{50}$ for $CD$, respectively, so the sum of all possible areas is $18 + 50 = 68$.

8. Let equilateral triangle $ABC$ with side length 6 be inscribed in a circle and let $P$ be on arc $AC$ such that $AP \cdot PC = 10$. Find the length of $BP$. 
9. In tetrahedron $ABCD$, $AB = 4$, $CD = 7$, and $AC = AD = BC = BD = 5$. Let $I_A$, $I_B$, $I_C$, and $I_D$ denote the incenters of the faces opposite vertices $A$, $B$, $C$, and $D$, respectively. It is provable that $AI_A$ intersects $BI_B$ at a point $X$, and $CI_C$ intersects $DI_D$ at a point $Y$. Compute $XY$.

Answer: $\sqrt{\frac{25}{2}}$

Solution 1: First, we make some preliminary observations. Let $M$ be the midpoint of $AB$ and $N$ be the midpoint of $CD$. We see that $I_A$ and $I_B$ lie on isosceles triangle $ABN$, since $AN$ and $BN$ are angle bisectors of $\angle CAD$ and $\angle CBD$, respectively. This shows that $AI_A$ and $BI_B$ are coplanar, so they intersect. Moreover, by symmetry, $X$ must lie on $MN$. Analogous facts hold for triangle $CDM$ and its associated points: in particular, $Y$ also lies on $MN$.

Now, we use mass points to determine the location of $X$ on $MN$. Let an ordered pair $(m, P)$ denote that point $P$ has mass $m$. Assume that masses $a$, $b$, $c$, and $d$ at points $A$, $B$, $C$, and $D$, respectively, are placed such that their sum lies at $X$ (that is, let $X$ be our fulcrum).

Since

$$(a + b + c + d, X) = (a, A) + ((b, B) + (c, C) + (d, D)),$$

it must be that

$$(b, B) + (c, C) + (d, D) = (b + c + d, I_A),$$

since $I_A$ is the unique point in the plane of $BCD$ and collinear with $X$ and $A$. This implies that $c = d$, since now $(c, C) + (d, D)$ must lie at the midpoint of $CD$, i.e. $N$. Now, since $X$ lies on $MN$, we know $(a, A) + (b, B)$ must lie at $M$, so $a = b$ as well. Finally, since $I_A$ lies on the angle bisector of $\angle BCD$, we know that if $CI_A$ is extended to intersect $BD$ at a point $Z$, then

$$\frac{BZ}{ZD} = \frac{BC}{CD} = \frac{5}{7} \implies \frac{b}{d} = \frac{7}{5}.$$ 

Hence, a suitable mass assignment is $a = b = 7$, $c = d = 5$. Now, we have that

$$(7, A) + (7, B)) + ((5, C) + (5, D)) = (14, M) + (10, N)$$

is at $X$, and so $MX = \frac{5}{12} MN$.

By similar logic, when we pick $Y$ to be the fulcrum, we get masses $a = b = 5$, $c = d = 4$, and so $MY = \frac{4}{3} MN$. Hence,

$$\frac{XY}{MN} = \frac{4}{9} \cdot \frac{5}{12} = \frac{1}{36}.$$

\[\text{\footnotesize For a rigorous introduction to mass points, we direct the interested reader to http://www.computing-wisdom.com/jstor/center_of_mass.pdf} \]
Finally, to compute \( MN \), we start by noting that

\[
CM = \sqrt{5^2 - 2^2} = \sqrt{21}
\]

by the Pythagorean Theorem in right triangle \( AMC \). Now, looking at right triangle \( MNC \), we get

\[
MN = \sqrt{21 - \left(\frac{7}{2}\right)^2} = \frac{\sqrt{35}}{2} \implies XY = \frac{\sqrt{35}}{72}.
\]

**Solution 2:** We present a variant of the first solution that does not require using mass points in three dimensions. Instead, we will use mass points on the triangle \( ABN \). Let \( X \) be our fulcrum. Recall that \( AXI_A \) are collinear. We need to compute \( BI_A = \frac{BI}{AI} \), which we can do by the Angle Bisector Theorem in triangle \( BCD \). Since \( CX_A \) bisects angle \( BCD \), we have

\[
BI_A = \frac{CB}{CN} = \frac{10}{7}.
\]

Therefore, we can assign a mass of 10 to \( N \) and 7 to \( A \). By symmetry, \( B \) also gets a mass of 7, so \( \frac{MX}{MN} = \frac{10}{7+7+10} = \frac{5}{12} \), as before. This computation extends to get \( \frac{MY}{MN} = \frac{4}{9} \).

Using these ratios, the final answer can be computed as in Solution 1.

10. Let triangle \( ABC \) have side lengths \( AB = 16, BC = 20, AC = 26 \). Let \( ACDE, ABFG, \) and \( BCHI \) be squares that are entirely outside of triangle \( ABC \). Let \( J \) be the midpoint of \( DG \), and \( L \) the midpoint of \( AC \). Find the area of triangle \( JKL \).

**Answer:** \( \frac{5\sqrt{1023}}{4} \)

**Solution:** We first prove a lemma. Let \( M \) be the midpoint of \( AB \) and \( N \) be the midpoint of \( EF \). Then \( KLMN \) is a square. We do this using vectors. Let \( v_1 = \overrightarrow{CA}, v_2 = \overrightarrow{BA}, u_1 = \overrightarrow{CD}, \) and \( u_2 = \overrightarrow{BF} \). We first calculate \( w = \overrightarrow{EF} \). Then \( w = (v_1 - v_2 + u_2) - (u_1 + v_1) = u_2 - v_2 - u_1 \).

Now, we calculate \( \overrightarrow{CN} \) in two different ways. First, \( \overrightarrow{CN} = v_1 + v_1 + \frac{w}{2} = v_1 + \frac{u_2 - v_2}{2} \). Second, \( \overrightarrow{CN} = v_1 - \frac{v_2}{2} + MN \). Equating these two gives us \( MN = \frac{u_2 + u_1}{2} \). Taking the dot product of \( MN \) with \( \overrightarrow{CB} = v_1 - v_2 \) gives \( v_1 \cdot v_2 - \frac{v_1 - v_2}{2} \), which is zero. In addition, note that \( u_1, u_2 \) are rotations of \( v_1, v_2 \) such that the angle between \( v_1 \) and \( v_2 \) is supplementary to the angle between \( u_1 \) and \( u_2 \). Hence, the length of \( MN \) is the same as the length of \( LM = \frac{v_1 - v_2}{2} \). A similar argument on \( LK \) gives the same result, and hence \( KLMN \) is a square.

Now, we see that \( LK = \frac{1}{2}BC \). Symmetrically, \( LJ = \frac{1}{2}AB \). Furthermore, angle \( KLJ \) is supplementary to angle \( ABC \). Hence, the area of triangle \( JKL \) is a quarter of the area of triangle \( ABC \), and so is the area of a triangle with side lengths half those of \( ABC \)'s. The area of \( JKL \) may thus be calculated with Heron’s formula:

\[
\sqrt{\frac{31}{2} \cdot \frac{15}{2} \cdot \frac{11}{2} \cdot \frac{5}{2}} = \frac{5\sqrt{1023}}{4}.
\]