1. In triangle ABC, $AC = 7$. D lies on AB such that $AD = BD = CD = 5$. Find BC.

   Answer: $\boldsymbol{\sqrt{51}}$

   Solution: Let $m\angle A = x$ and $m\angle B = y$. Note that we have two pairs of isosceles triangles, so $m\angle A = m\angle ACD$ and $m\angle B = m\angle BCD$. Since $m\angle ACD + m\angle BCD = m\angle ACB$, we have

$$180^\circ = m\angle A + m\angle B + m\angle ACB = 2x + 2y \implies m\angle ACB = x + y = 90^\circ.$$

   Since $\angle ACB$ is right, we can use the Pythagorean Theorem to compute BC as

$$\sqrt{10^2 - 7^2} = \boxed{\sqrt{51}}.$$

   For a shortcut, note that D is the circumcenter of ABC and lies on the triangle itself, so it must lie opposite a right angle.

2. What is the perimeter of a rectangle of area 32 inscribed in a circle of radius 4?

   Answer: $\boldsymbol{16\sqrt{2}}$

   Solution: It turns out the rectangle is actually a square with side length $4\sqrt{2}$, and hence has perimeter $\boxed{16\sqrt{2}}$.

3. Robin has obtained a circular pizza with radius 2. However, being rebellious, instead of slicing the pizza radially, he decides to slice the pizza into 4 strips of equal width both vertically and horizontally. What is the area of the smallest piece of pizza?

   Answer: $\boldsymbol{\frac{\pi}{3} + 1 - \sqrt{3}}$

   Solution 1: Let O be the center of the circle, and let A and B lie on the circle such that $m\angle AOB = 90^\circ$. Call M the midpoint of AO and N the midpoint of BO. Let C lie on minor arc AB such that $CM \perp OA$, and let D lie on minor arc AB such that $DN \perp OB$. Finally, let CM and DN intersect at E. Now, the problem is to find the area of the region bounded by DE, EC, and minor arc CD.

   Notice that $ON = 1$ and $OD = 2$, so OND is a 30-60-90 right triangle. Since DN and AO are parallel, $m\angle NDO = m\angle AOD = 30^\circ$. We now see that the area of the region bounded by AM, ME, ED, and arc DA can be expressed as the sum of the areas of triangle OND and sector AOD minus the area of square $MONE$, which evaluates to

$$\frac{1}{2} \cdot 1 \cdot \sqrt{3} + \frac{\pi \cdot 2^2}{12} - 1 = \frac{\sqrt{3}}{2} + \frac{\pi}{3} - 1.$$

   Finally, let x denote the desired area. Then, the area of sector AOB is

$$1 + 2\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} - 1\right) + x = \frac{\pi \cdot 2^2}{4} \implies x = \boxed{\frac{\pi}{3} + 1 - \sqrt{3}}.$$

   Solution 2: When the pizza is sliced 4 times in both directions, the result is 4 unit squares, 8 congruent approximate quadrilaterals (one edge is curved), and 4 congruent approximate triangles (again, one edge is curved). Call the area of an approximate quadrilateral x and an approximate triangle y. Since all these pieces form a circle of radius 2, we get

$$8x + 4y = 4\pi - 4.$$

Now, consider the long horizontal slice at the bottom of the pizza, consisting of 2 approximate quadrilaterals and 2 approximate triangles. Define the endpoints of the slice to be A and B. Define the center of the pizza to be C. Consider the sector of the pizza cut out by AC and BC. This is one third of the pizza, as $\angle ACB = 120°$, and $\angle ABC = \angle BAC = 30°$. Therefore, the area of the sector is $4\pi/3$ and the area of triangle ABC is $\sqrt{3}$. Hence, we get

$$2x + 2y = \frac{4\pi}{3} - \sqrt{3}.$$

Therefore, we have the system

$$\begin{aligned} 2x + y &= \pi - 1 \\ 2x + 2y &= \frac{4\pi}{3} - \sqrt{3}. \end{aligned}$$

Solving this system gives

$$\begin{aligned} x &= \frac{\pi}{3} - 1 + \frac{\sqrt{3}}{2} \\ y &= \frac{\pi}{3} + 1 - \sqrt{3}. \end{aligned}$$

Therefore, the smallest piece of pizza has area

$$\boxed{\frac{\pi}{3} + 1 - \sqrt{3}}.$$

4. $ABCD$ is a regular tetrahedron with side length 1. Find the area of the cross section of $ABCD$ cut by the plane that passes through the midpoints of AB, AC, and CD.

Answer: $\frac{1}{4}$

Solution: First, note that the plane also passes through the midpoint of BD by symmetry, e.g. across the plane containing AD perpendicular to BC. Let M, N, O, and P denote the midpoints of BA, AC, CD, and DB, respectively. $MN = NO = OP = PM = \frac{1}{2}$ because they are all midlines of faces of the tetrahedron. Hence, the cross section is a rhombus. Furthermore, $MO \cong NP$ because both equal the distance between midpoints of opposite sides (alternatively, this congruence can be demonstrated by rotating $ABCD$ such that N and P coincide with the previous locations of M and O). Hence, $MNOP$ is a square, and its area is $\left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{4}}$.

5. In square $ABCD$ with side length 2, let P and Q both be on side AB such that $AP = BQ = \frac{1}{2}$. Let E be a point on the edge of the square that maximizes the angle PEQ. Find the area of triangle PEQ.

Answer: $\frac{\sqrt{3}}{4}$

Solution: For any choice of E, we can draw the circumcircle of PEQ. Angle PEQ is inscribed inside the minor arc of chord PQ, which is of constant length (it must always be the minor arc because PEQ is clearly always acute). Therefore, maximizing $m\angle PEQ$ is equivalent to maximizing the measure of minor arc PQ, which in turn is equivalent to minimizing the radius of the circle.

Hence, we wish to find the smallest circle that intersects $ABCD$ at P, Q, and at least one other point. A circle of radius 1 can be tangent to sides BC and AD, while a circle with a smaller radius clearly cannot touch any of the sides of the square. Hence, it is this circle we desire. Let this circle be centered at O. OPQ is equilateral, so the height from O to PQ has length $\frac{\sqrt{3}}{2}$. This is also the height from the points of tangency on AD or BC to PQ. E may be either one of these points, resulting in PEQ having area $\boxed{\frac{\sqrt{3}}{4}}$.

6. $ABCD$ is a rectangle with $AB = CD = 2$. A circle centered at O is tangent to BC, CD, and AD (and hence has radius 1). Another circle, centered at P, is tangent to circle O at point T and is also tangent to AB and BC. If line AT is tangent to both circles at T, find the radius of circle P.

Answer: $\mathbf{\frac{3-\sqrt{5}}{2}}$

Solution: Let the radius of circle P be r. Draw OP, noting that it is perpendicular to AT at T. Let Q be the point of tangency between circle O and AD. If we drop a perpendicular from P to meet OQ (extended) at R, then we know that $OR = 1 - r$ and $OP = 1 + r$, so by the Pythagorean theorem, $PR = 2\sqrt{r}$. Thus, $AQ = 2\sqrt{r} + r$.

Let AB be tangent to P at U. By the Two-Tangent Theorem, $AQ \cong AT \cong AU$. Since $UB = r$, we have

$$(2\sqrt{r} + r) + r = 2 \implies r = \boxed{\frac{3-\sqrt{5}}{2}}.$$

7. $ABCD$ is a square such that $\overline{AB}$ lies on the line $y = x + 4$ and points C and D lie on the graph of parabola $y^2 = x$. Compute the sum of all possible areas of $ABCD$.

Answer: 68

Solution 1: First, shift the coordinate system so that the line goes through the origin and the parabola is now at $x = y^2 + 4$.

Let CD lie on the line $y = x + b$. The distance between lines AB and CD is therefore $\frac{|b|}{\sqrt{2}}$, which can be proven by drawing 45-45-90 triangles. This distance is precisely $AD = BC$, so CD must also have this length. Hence, the y-coordinates of C and D must have difference $\frac{|b|}{2}$, again by 45-45-90 triangles.

Substituting $x = y - b$ to $x = y^2 + 4$ yields $y^2 - y + (b + 4) = 0$. The difference between two solutions is $\sqrt{1 - 4(b + 4)} = \frac{|b|}{2}$, which simplifies to $b^2 + 16b + 60 = 0$. The area of $ABCD$ is $\frac{1}{2}b^2$, so we want $\frac{1}{2}$ times the square of the possible values of b as our answer. We can compute this as $\frac{16^2 - 2\cdot 60}{2} = \boxed{68}$.

Solution 2: Let $C = (y_1^2, y_1)$ and $D = (y_2^2, y_2)$, and assume without loss of generality that the points are positioned such that $y_1 < y_2$. Viewing this in the complex plane, we have $B - C = (D - C)i$, so $B = (y_1^2 + y_1 - y_2, y_2^2 - y_1^2 + y_1)$. Plugging this into $y = x + 4$ gives us $y_2^2 - 2y_1^2 + y_2 - 4 = 0$. Since $\overline{AB} \parallel \overline{DC}$, the slope of $\overline{DC}$ is 1, so $\frac{y_1 - y_2}{y_1^2 - y_2^2} = 1 \implies y_1 + y_2 = 1$. Solving this system of equations gives us two pairs of solutions for (y_1, y_2), namely $(-1, 2)$ and $(-2, 3)$. These give $\sqrt{18}$ and $\sqrt{50}$ for CD, respectively, so the sum of all possible areas is $18 + 50 = \boxed{68}$.

8. Let equilateral triangle ABC with side length 6 be inscribed in a circle and let P be on arc AC such that $AP \cdot PC = 10$. Find the length of BP.

Answer: $\sqrt{46}$

Solution: Note that $BP = AP + CP$. To prove this, form equilateral triangle APD where D lies on the extension of CP. Then triangle ACD is congruent to triangle ABP (and can obtained by rotating triangle ABP by 60 degrees). Therefore, $CD = AP + PC = BP$. Alternatively, apply Ptolemy's Theorem to cyclic quadrilateral $ABCP$, which gives $BP = AP + CP$ directly.

Next, apply the Law of Cosines on triangle APC to deduce that $AP^2 + CP^2 + AP \cdot CP = 6^2$ (we have used the fact that $m\angle APC = 120°$, since it is opposite the $60°$ angle ABC). Hence, $(AP + CP)^2 = 36 + 10$ so $BP = AP + CP = \boxed{\sqrt{46}}$.

9. In tetrahedron $ABCD$, $AB = 4$, $CD = 7$, and $AC = AD = BC = BD = 5$. Let I_A, I_B, I_C, and I_D denote the incenters of the faces opposite vertices A, B, C, and D, respecitvely. It is provable that AI_A intersects BI_B at a point X, and CI_C intersects DI_D at a point Y. Compute XY.

Answer: $\frac{\sqrt{35}}{72}$

Solution 1: First, we make some preliminary observations. Let M be the midpoint of AB and N be the midpoint of CD. We see that I_A and I_B lie on isosceles triangle ABN, since AN and BN are angle bisectors of $\angle CAD$ and $\angle CBD$, respectively. This shows that AI_A and BI_B are coplanar, so they intersect. Moreover, by symmetry, X must lie on MN. Analogous facts hold for triangle CDM and its associated points: in particular, Y also lies on MN.

Now, we use mass points to determine the location of X on MN[1]. Let an ordered pair (m, P) denote that point P has mass m. Assume that masses a, b, c, and d at points A, B, C, and D, respectively, are placed such that their sum lies at X (that is, let X be our fulcrum).

Since

$$(a + b + c + d, X) = (a, A) + ((b, B) + (c, C) + (d, D)),$$

it must be that

$$(b, B) + (c, C) + (d, D) = (b + c + d, I_A),$$

since I_A is the unique point in the plane of BCD and collinear with X and A. This implies that $c = d$, since now $(c, C) + (d, D)$ must lie at the midpoint of CD, i.e. N. Now, since X lies on MN, we know $(a, A) + (b, B)$ must lie at M, so $a = b$ as well. Finally, since I_A lies on the angle bisector of $\angle BCD$, we know that if CI_A is extended to intersect BD at a point Z, then

$$\frac{BZ}{ZD} = \frac{BC}{CD} = \frac{5}{7} \implies \frac{b}{d} = \frac{7}{5}.$$

Hence, a suitable mass assignment is $a = b = 7$, $c = d = 5$. Now, we have that

$$((7, A) + (7, B)) + ((5, C) + (5, D)) = (14, M) + (10, N)$$

is at X, and so $MX = \frac{5}{12} MN$.

By similar logic, when we pick Y to be the fulcrum, we get masses $a = b = 5$, $c = d = 4$, and so $MY = \frac{4}{9} MN$. Hence,

$$\frac{XY}{MN} = \frac{4}{9} - \frac{5}{12} = \frac{1}{36}.$$

[1]For a rigorous introduction to mass points, we direct the interested reader to `http://www.computing-wisdom.com/jstor/center_of_mass.pdf`

Finally, to compute MN, we start by noting that

$$CM = \sqrt{5^2 - 2^2} = \sqrt{21}$$

by the Pythagorean Theorem in right triangle AMC. Now, looking at right triangle MNC, we get

$$MN = \sqrt{21 - \left(\frac{7}{2}\right)^2} = \frac{\sqrt{35}}{2} \implies XY = \boxed{\frac{\sqrt{35}}{72}}.$$

Solution 2: We present a variant of the first solution that does not require using mass points in three dimensions. Instead, we will use mass points on the triangle ABN. Let X be our fulcrum. Recall that AXI_A are colinear. We need to compute $\frac{BI_A}{I_AN}$, which we can do by the Angle Bisector Theorem in triangle BCD. Since CX_A bisects angle BCD, we have $\frac{BI_A}{I_AN} = \frac{CB}{CN} = \frac{10}{7}$. Therefore, we can assign a mass of 10 to N and 7 to A. By symmetry, B also gets a mass of 7, so $\frac{MX}{MN} = \frac{10}{7+7+10} = \frac{5}{12}$, as before. This computation extends to get $\frac{MY}{MN} = \frac{4}{9}$.

Using these ratios, the final answer can be computed as in Solution 1.

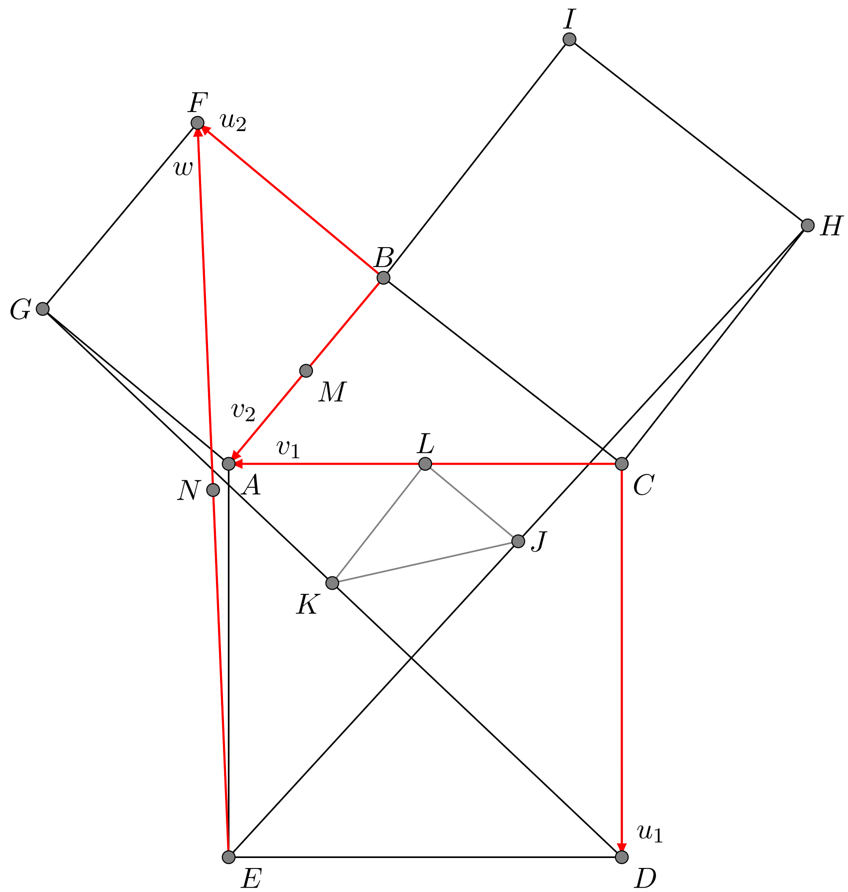
10. Let triangle ABC have side lengths $AB = 16$, $BC = 20$, $AC = 26$. Let $ACDE$, $ABFG$, and $BCHI$ be squares that are entirely outside of triangle ABC. Let J be the midpoint of EH, K be the midpoint of DG, and L the midpoint of AC. Find the area of triangle JKL.

Answer: $\frac{5\sqrt{1023}}{4}$

Solution: We first prove a lemma. Let M be the midpoint of AB and N be the midpoint of EF. Then $KLMN$ is a square. We do this using vectors. Let $v_1 = \overrightarrow{CA}$, $v_2 = \overrightarrow{BA}$, $u_1 = \overrightarrow{CD}$, and $u_2 = \overrightarrow{BF}$. We first calculate $w = \overrightarrow{EF}$. Then $w = (v_1 - v_2 + u_2) - (u_1 + v_1) = u_2 - v_2 - u_1$. Now, we calculate $\overrightarrow{CN}$ in two different ways. First, $\overrightarrow{CN} = u_1 + v_1 + \frac{w}{2} = v_1 + \frac{u_2}{2} + \frac{u_1}{2} - \frac{v_2}{2}$. Second, $\overrightarrow{CN} = v_1 - \frac{v_2}{2} + \overrightarrow{MN}$. Equating these two gives us $\overrightarrow{MN} = \frac{u_2+u_1}{2}$. Taking the dot product of $\overrightarrow{MN}$ with $\overrightarrow{CB} = v_1 - v_2$ gives $\frac{v_1 \cdot u_2 - v_2 \cdot u_1}{2}$, which is zero. In addition, note that u_1, u_2 are rotations of v_1, v_2 such that the angle between v_1 and v_2 is supplementary to the angle between u_1 and u_2. Hence, the length of $\overrightarrow{MN}$ is the same as the length of $\overrightarrow{LM} = \frac{v_1 - v_2}{2}$. A similar argument on $\overrightarrow{LK}$ gives the same result, and hence $KLMN$ is a square.

Now, we see that $LK = \frac{1}{2}BC$. Symmetrically, $LJ = \frac{1}{2}AB$. Furthermore, angle KLJ is supplementary to angle ABC. Hence, the area of triangle JKL is a quarter of the area of triangle ABC, and so is the area of a triangle with side lengths half those of ABC's. The area of JKL may thus be calculated with Heron's formula:

$$\sqrt{\frac{31}{2} \cdot \frac{15}{2} \cdot \frac{11}{2} \cdot \frac{5}{2}} = \boxed{\frac{5\sqrt{1023}}{4}}.$$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