1. Compute $\displaystyle\int_{e^{e^e}}^{\infty} \frac{dx}{x(\log x)(\log\log x)(\log\log\log x)^{4/3}}$.

Answer: 3

Solution: Substitute $u = \log\log\log x$, yielding

$$\int_1^{\infty} u^{-\frac{4}{3}}\,du = \left[-3u^{-\frac{1}{3}}\right]_1^{\infty} = \boxed{3}.$$

2. For which integers $1 \le m \le 10$ is it true that

$$\int_0^{\pi} (\cos x)(\cos 2x)\cdots(\cos mx)\,dx = 0?$$

Answer: 1, 2, 5, 6, 9, 10

Solution 1: Apply the product-to-sum formula repeatedly on the integrand, until we have no more products of cosines. What remains in the integrand (besides some factors of two which we can ignore) are 2^m terms, each written as $\cos((\pm 1 \pm 2 \pm \cdots \pm m)x)$. Each such term integrates to 0 if and only if $(\pm 1 \pm \cdots \pm m)$ sums to a nonzero number; otherwise, the term is $\cos 0 = 1$, which integrates to something positive.

Therefore, the integral is nonzero if and only if there is some choice for the $\pm$ such that we can get $\pm 1 \pm \cdots \pm m = 0$.

Observe that if $m \equiv 1, 2 \bmod 4$ (i.e. the numbers $1, \ldots, m$ contains an odd number of odd numbers), then $\pm 1 \pm \cdots \pm m$ is always odd and hence nonzero; therefore, the integral is zero when $m \equiv 1, 2 \bmod 4$.

When $m \equiv 0 \bmod 4$, we can write $m = 4k$, so that $(1 + 4k) - (2 + (4k - 1)) + \cdots = 0$. Therefore the integral is nonzero.

When $m \equiv 3 \bmod 4$, we can write $m = 4k+3$, so $(1+2-3)+(4+(4k+3))-(5+(4k+2))+\cdots = 0$. Therefore the integral is nonzero.

This covers all of the cases, so we can conclude that the above integral is zero if and only if $m \equiv 1, 2 \bmod 4$. Therefore, we should report $\boxed{1, 2, 5, 6, 9, 10}$.

Solution 2: Observe that for all k, we have that

$$\begin{aligned} \cos((2k+1)(\pi - x)) &= \cos((2k+1)\pi - (2k+1)x) = \cos(\pi - (2k+1)x) = -\cos((2k+1)x) \\ \cos(2k(\pi - x)) &= \cos(2k\pi - 2kx) = \cos(-2kx) = \cos(2kx). \end{aligned}$$

This means that $\cos(nx)$ is odd about the line $x = \frac{\pi}{2}$ when n is odd, and even about the line $x = \frac{\pi}{2}$ when n is even.

Let $f_m(x) = (\cos x)(\cos 2x)\cdots(\cos mx)$. Then

$$f_m(\pi - x) = \begin{cases} f_m(x) & m \equiv 0, 3 \bmod 4 \\ -f_m(x) & m \equiv 1, 2 \bmod 4. \end{cases}$$

Therefore, by symmetry, $\int_0^{\pi} f_m(x)\,dx = 0$ when $m \equiv 1, 2 \bmod 4$.

It remains to show that $\int_0^{\pi} f_m(x)\,dx \neq 0$ when $m \equiv 0, 3 \bmod 4$. There are a number of ways to do this, such as the solution above.

3. Let $f(x) = \sum_{n=0}^{\infty} \frac{\sin nx}{n!}$. Compute $f\left(\frac{\pi}{3}\right)$.

Answer: $\boldsymbol{e^{1/2} \sin\left(\frac{\sqrt{3}}{2}\right)}$

Solution:

$$\sum_{n=0}^{\infty} \frac{\sin nx}{n} = \text{Im}\left(\sum_{n=0}^{\infty} \frac{e^{inx}}{n!}\right) = \text{Im}\left(e^{e^{ix}}\right) = \text{Im}\left(e^{\cos x + i \sin x}\right) = e^{\cos x} \cdot \text{Im}(e^{i \sin x}) = e^{\cos x} \sin \sin x.$$

Therefore $f(x) = e^{\cos x} \sin \sin x$ and the answer follows.