1. Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

   Answer: 15

   Solution: Let d be the length of one lap in miles. Then he needs to complete the four laps in $\frac{4d}{10} = \frac{2d}{5}$ hours. He has already spent $\frac{3d}{9} = \frac{d}{3}$ hours on the first three laps, so he has $\frac{2d}{5} - \frac{d}{3} = \frac{d}{15}$ hours left. Therefore, he must maintain a speed of $\boxed{15}$ mph on the final lap.

2. A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

   Answer: 10/11

   Solution: After removing x from 10, and then increasing that amount by 10%, we must end up with at least the amount we started with, 10 pounds. That is, the maximum value of x must satisfy $\frac{11}{10}(10 - x) = 10$. Solving for x, we get that $x = \boxed{10/11}$.

3. Karl likes the number 17. His favorite polynomials are monic quadratics with integer coefficients such that 17 is a root of the quadratic and the roots differ by no more than 17. Compute the sum of the coefficients of all of Karl's favorite polynomials. (A monic quadratic is a quadratic polynomial whose x^2 term has a coefficient of 1.)

   Answer: 8960

   Solution: All of Karl's favorite quadratics take the form $(x - r)(x - 17)$, where $0 \leq r \leq 34$. The sum of the coefficients of any polynomial can be determined by evaluating the polynomial at $x = 1$. This gives $16r - 16$. Then $\sum_{r=0}^{34}(16r - 16) = 16 \cdot \frac{34 \cdot 35}{2} - 16 \cdot 35 = \boxed{8960}$.

4. Given that $f(x) + 2f(8 - x) = x^2$ for all real x, compute $f(2)$.

   Answer: 68/3

   Solution: Substituting $x = 2$, we get that $f(2) + 2f(6) = 4$. Substituting $x = 6$, we get that $f(6) + 2f(2) = 36$. Solving for $f(2)$ and $f(6)$ gives us that $f(6) = -28/3$ and $f(2) = \boxed{68/3}$.

5. For exactly two real values of b, b_1 and b_2, the line $y = bx - 17$ intersects the parabola $y = x^2 + 2x + 3$ at exactly one point. Compute $b_1^2 + b_2^2$.

   Answer: 168

   Solution: We have that b is a valid number if and only if $(x^2+2x+3)-(bx-17) = x^2+(2-b)x+20$ has exactly one real root. This means that $2 - b = \pm 2\sqrt{20}$, so $b = 2 \pm 2\sqrt{20}$. $b_1^2 + b_2^2$ is therefore $2(2^2) + 2(2\sqrt{20})^2 = 8 + 160 = \boxed{168}$.

6. Compute the largest root of $x^4 - x^3 - 5x^2 + 2x + 6$.

   Answer: $\mathbf{\frac{1+\sqrt{13}}{2}}$

   Solution: Note that $x^4-x^3-5x^2+2x+6 = (x^4-5x^2+6)-x(x^2-2) = (x^2-2)(x^2-3)-x(x^2-2) = (x^2 - 2)(x^2 - x - 3)$. The two largest candidate roots are therefore $\sqrt{2}$ and $\frac{1 + \sqrt{13}}{2}$. Note that $\sqrt{13} > 3$, so $\frac{1 + \sqrt{13}}{2} > 2 > \sqrt{2}$, so therefore the largest root is $\boxed{\frac{1 + \sqrt{13}}{2}}$.

7. Find all real x that satisfy $\sqrt[3]{20x + \sqrt[3]{20x + 13}} = 13$.

Answer: 546/5

Solution: Observe that $f(a) = \sqrt[3]{20x + a}$ is an increasing function in a, so the only way that $f(f(a)) = a$ can be true is if $f(a) = a$. Solving $\sqrt[3]{20x + 13} = 13$, we obtain $x = \boxed{546/5}$.

8. Find the sum of all real x such that

$$\frac{4x^2 + 15x + 17}{x^2 + 4x + 12} = \frac{5x^2 + 16x + 18}{2x^2 + 5x + 13}.$$

Answer: $-11/3$

Solution: Let $f(x) = 4x^2 + 15x + 17$, $g(x) = x^2 + 4x + 12$, and $h(x) = x^2 + x + 1$. Then, the given equation becomes

$$\begin{aligned} &\frac{f(x)}{g(x)} = \frac{f(x) + h(x)}{g(x) + h(x)} \\ &\implies f(x)g(x) + f(x)h(x) = f(x)g(x) + g(x)h(x) \\ &\implies f(x)h(x) = g(x)h(x). \end{aligned}$$

Since $h(x) > 0$ for all real x, we may divide through by $h(x)$ to get

$$\begin{aligned} &f(x) = g(x) \\ &\implies 4x^2 + 15x + 17 = x^2 + 4x + 12 \\ &\implies 3x^2 + 11x + 5 = 0. \end{aligned}$$

The discriminant of this quadratic is

$$11^2 - 4 \cdot 3 \cdot 5 = 61 > 0,$$

so it has two real roots. By Vieta's, the sum of these roots is $\boxed{-11/3}$.

9. Let $a = -\sqrt{3} + \sqrt{5} + \sqrt{7}$, $b = \sqrt{3} - \sqrt{5} + \sqrt{7}$, $c = \sqrt{3} + \sqrt{5} - \sqrt{7}$. Evaluate

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}.$$

Answer: 30

Solution: Putting everything over a common denominator, we can rewrite the expression as

$$\frac{a^4(b-c) - b^4(a-c) + c^4(a-b)}{(a-b)(a-c)(b-c)} = \frac{a^4b - ab^4 - a^4c + ac^4 + b^4c - bc^4}{(a-b)(a-c)(b-c)}.$$

Notice that if $a = b$, the numerator becomes $a^5 - a^5 - a^4c + ac^4 + a^4c - ac^4 = 0$; similarly if $a = c$ or $b = c$. This means that the numerator is in fact divisible by $(a-b)(a-c)(b-c)$. Factoring, we find that the above expression is equal to

$$\frac{(a-b)(b-c)(a-c)(a^2 + b^2 + c^2 + ab + bc + ac)}{(a-b)(b-c)(a-c)} = a^2 + b^2 + c^2 + ab + bc + ac$$

as long as the original expression was well-defined. But we have

$$a^2 + b^2 + c^2 + ab + bc + ac = \frac{1}{2}\left((a+b)^2 + (b+c)^2 + (c+a)^2\right)$$

and plugging in the given values of a, b, c gives

$$\frac{1}{2}\left((2\sqrt{7})^2 + (2\sqrt{3})^2 + (2\sqrt{5})^2\right) = 2(7+3+5) = \boxed{30}.$$

10. Given a complex number z such that $z^{13} = 1$, find all possible values of $z+z^3+z^4+z^9+z^{10}+z^{12}$.

Answer: 6, $\frac{-1\pm\sqrt{13}}{2}$

Solution: First of all, if $z = 1$, then the expression is simply equal to $\boxed{6}$. Otherwise, let $\omega = z + z^3 + z^4 + z^9 + z^{10} + z^{12}$. We find that

$$\omega^2 = z^2+z^6+z^8+z^5+z^7+z^{11}+2(z^4+z^5+z^{10}+z^{11}+1+z^7+z^{12}+1+z^2+1+z+z^3+z^6+z^8+z^9).$$

Applying the identity $z + z^2 + z^3 + \cdots + z^{12} = -1$, we arrive at $\omega^2 = -1 - \omega + 2(3-1) = 3 - \omega$, and the solutions to the quadratic are $\omega = \boxed{\dfrac{-1 \pm \sqrt{13}}{2}}$.