1. A circle with radius 1 has diameter AB. C lies on this circle such that $\overset{\frown}{AC}$ / $\overset{\frown}{BC}$= 4. $\overline{AC}$ divides the circle into two parts, and we will label the smaller part Region I. Similarly, $\overline{BC}$ also divides the circle into two parts, and we will denote the smaller one as Region II. Find the positive difference between the areas of Regions I and II.

   Answer: $\frac{3\pi}{10}$

   Solution: Let O be the center of the circle. Note that CO bisects AB, so the areas of $\triangle ACO$ and $\triangle BCO$ are equal. Hence, the desired difference in segment areas is equal to the difference in the areas of the corresponding sectors. The sector corresponding to $\overset{\frown}{AC}$ has area $\frac{2\pi}{5}$, and the sector corresponding to $\overset{\frown}{BC}$ has area $\frac{\pi}{10}$, so the desired difference is $\boxed{\frac{3\pi}{10}}$.

2. In trapezoid $ABCD$, $BC \parallel AD$, $AB = 13$, $BC = 15$, $CD = 14$, and $DA = 30$. Find the area of $ABCD$.

   Answer: 252

   Solution: We can use the standard method of setting up a two-variable system and solving for the height of the trapezoid. However, since one base is half the length of the other, we may take a shortcut. Extend AB and CD until they meet at E. Clearly, BC is a midline of triangle EAD, so we have $EA = 2BA = 26$ and $ED = 2CD = 28$. The area of EAD is therefore four times that of a standard 13-14-15 triangle, which we know is $\frac{1}{2} \cdot 14 \cdot 12 = 84$ (since the altitude to the side of length 14 splits the triangle into 9-12-15 and 5-12-13 right triangles). The area of the trapezoid is $\frac{3}{4}$ the area of EAD by similar triangles, and is therefore $3 \cdot 84 = \boxed{252}$.

   A similar solution draws lines from B and C to the midpoint of AD to form three $13-14-15$ triangles.

3. Let ABC be an equilateral triangle with side length 1. Draw three circles O_a, O_b, and O_c with diameters BC, CA, and AB, respectively. Let S_a denote the area of the region inside O_a and outside of O_b and O_c. Define S_b and S_c similarly, and let S be the area of the region inside all three circles. Find $S_a + S_b + S_c - S$.

   Answer: $\frac{\sqrt{3}}{2}$

   Solution: Let x be 1/4 the area of ABC, and let y be the area of a 60 degree sector of O_a minus x. Note that

   $$S_a = S_b = S_c = 3x + y, \quad S = x + 3y,$$

   so $S_a + S_b + S_c - S = 8x = 2|\triangle ABC| = \boxed{\sqrt{3}/2}$.

4. Let $ABCD$ be a rectangle with area 2012. There exist points E on AB and F on CD such that $DE = EF = FB$. Diagonal AC intersects DE at X and EF at Y. Compute the area of triangle EXY.

   Answer: $\frac{503}{6}$

   Solution: Let (XYZ) denote the area of triangle XYZ.

   After a bit of angle-chasing, we can use SAS congruence to prove that $\triangle DEF \cong \triangle BFE$, so $EB \cong DF$ and therefore $AE \cong FC$. If we draw altitudes from E and F onto CD and AB, respectively, we note that $2AE = 2FC = DF = BE$, so $AE = \frac{1}{3}AB$.

   Next, note that $\triangle AEX \sim \triangle CDX$, so $\frac{CX}{AX} = \frac{CD}{AE} = 3$. Also, $\triangle CFY \sim \triangle AEY$, so $\frac{CY}{AY} = \frac{CF}{AE} = 1$. Hence, $XY = \frac{1}{4}AC \implies (EXY) = \frac{1}{4}(EAC)$.

Finally, $AE = \frac{1}{3}AB \implies (EAC) = \frac{1}{3}(BAC)$. Since BAC is half the rectangle and therefore has area 1006, we get $(EXY) = \frac{1006}{12} = \boxed{\dfrac{503}{6}}$.

5. What is the radius of the largest sphere that fits inside an octahedron of side length 1?

Answer: $\mathbf{\frac{1}{\sqrt{6}}}$

Solution: It is obvious that the sphere must be tangent to each face, because if not, then it can be moved so that it is tangent to four faces; now the radius can be increased until the sphere is tangent to the other four. Additionally, it is clear that the center of the sphere should be in the center of the octahedron.

Now notice that the sphere must be tangent to the octahedron at the centroid of each face. This can be seen by symmetry. It is clear that it should be tangent somewhere along the median from one vertex to the opposite side, and this is true for all three medians, which meet at the centroid.

Now we can proceed in a few ways. One way is to isolate one half of the octahedron i.e. a square-based pyramid. Slice this pyramid in half perpendicular to the square base and parallel to one of the sides of the square base. This slice will go through the medians of two opposite triangular faces, in addition to the center of the sphere itself. Hence, we get an isosceles triangle ABC with base $BC = 1$ and legs of length $\sqrt{3}/2$. O, the center of the sphere, is the midpoint of BC. The radius of the sphere is the altitude from O to AB. If this altitude intersects AB at D, then we have

$$OD \cdot AB = AO \cdot BO,$$

since both equal twice the area of AOB, and so $DO = \frac{AO \cdot BO}{AB} = \frac{(1/\sqrt{2})(1/2)}{\sqrt{3}/2} = \boxed{1/\sqrt{6}}$.

Alternatively, note that our octahedron can be obtained by reflecting the region $x+y+z \leq 1/\sqrt{2}$, $x, y, z \geq 0$ across the xy, yz, and zx planes. The inscribing sphere has its center at origin, so its radius is the distance from the origin to the plane $x + y + z = 1/\sqrt{2}$, which is $1/\sqrt{6}$.

6. A red unit cube $ABCDEFGH$ (with E below A, F below B, etc.) is pushed into the corner of a room with vertex E not visible, so that faces $ABFE$ and $ADHE$ are adjacent to the wall and face $EFGH$ is adjacent to the floor. A string of length 2 is dipped in black paint, and one of its endpoints is attached to vertex A. How much surface area on the three visible faces of the cube can be painted black by sweeping the string over it?

Answer: $\mathbf{\frac{2\pi}{3} + \sqrt{3} - 1}$

Solution: First, it is clear that all of face $ABCD$ can be painted black. This has area 1.

Now we look at the other two visible faces. By symmetry, we only need to consider one of these faces, say $BCGF$. Unfold $BCGF$ along BC so that it is coplanar with $ABCD$, forming a rectangle $AF'G'D$ with width 1 and height 2. Now, it is clear that the region that can be painted on $BCGF$ is precisely the part of $BCG'F'$ that is at most two units away from A. Let a circle centered at A with radius two intersect DG' at X. Since $AX = 2$, $AD = 1$, and $AD \perp XD$, we conclude that $m\angle DAX = \frac{\pi}{3} \implies \angle F'AX = \frac{\pi}{6}$. Letting $(P_1P_2 \ldots P_n)$ denote the area of the n-gon with vertices $P_1, \ldots, P_n$, we can write the desired area as

$$\text{area of sector } F'AX + (AXD) - (ABCD) = \frac{2^2\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

Putting all this together, we get our final answer to be

$$1 + 2\left(\pi/3 + \frac{\sqrt{3}}{2} - 1\right) = \boxed{\frac{2\pi}{3} + \sqrt{3} - 1}.$$

7. Let ABC be a triangle with incircle O and side lengths 5, 8, and 9. Consider the other tangent line to O parallel to BC, which intersects AB at B_a and AC at C_a. Let r_a be the inradius of triangle AB_aC_a, and define r_b and r_c similarly. Find $r_a + r_b + r_c$.

   Answer: $\frac{6\sqrt{11}}{11}$

   Solution: We claim that the answer is equal to the inradius in general. Let $T_a = AB_aC_a$, $T_b = A_bBC_b$, $T_c = A_cB_cC$ be the smaller triangles cut by the tangents drawn to O. Also let D, E, and F be the points of tangency between O and BC, CA, and AB respectively. By considering the fact that tangents to O from the same point should have the same length, we have $AB_a + B_aC_a + C_aA = AE + AF$. If we sum this over all vertices, then we can see that the sum of the perimeters of T_a, T_b, and T_c equals the perimeter of A. Then, the Principle of Similarity gives $r_a + r_b + r_c = r$ where r is inradius of ABC. The inradius can be calcultaed by Heron's Formula as

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\sqrt{11 \cdot 6 \cdot 3 \cdot 2}}{11} = \boxed{\frac{6\sqrt{11}}{11}}.$$

   Alternatively, let h_a denote the height of the altitude from A to BC, and let r be the inradius of ABC. Since $\triangle ABC \sim \triangle AB_aC_a$ and since the altitude from A to B_aC_a has length $h_a - 2r$, we get

$$\frac{r_a}{r} = \frac{h_a - 2r}{h_a}.$$

   Noticing that

$$r = \frac{(ABC)}{\frac{1}{2}(a+b+c)} = \frac{ah_a}{a+b+c},$$

   we get

$$r_a = r - 2r\frac{a}{a+b+c}.$$

   Applying the same reasoning to r_a and r_b, we can compute

$$r_a + r_b + r_c = 3r - 2r = r.$$

8. Let ABC be a triangle with side lengths 5, 6, and 7. Choose a radius r and three points outside the triangle O_a, O_b, and O_c, and draw three circles with radius r centered at these three points. If circles O_a and O_b intersect at C, O_b and O_c intersect at A, O_c and O_a intersect at B, and all three circles intersect at a fourth point, find r.

   Answer: $\frac{35\sqrt{6}}{24}$

   Solution: Let the point where all three circles intersect be denoted as X.

   First, note that AO_bXO_c, BO_cXO_a, and CO_aXO_b are all rhombi. This helps us easily prove that $AO_b \parallel BO_a$. Since these segments are also congruent, we get that ABO_aO_b is a parellogram, and hence $AB \cong O_aO_b$. Similarly, $BC \cong O_bO_c$ and $CA \cong O_cO_a$.

Since r is clearly the circumradius of $O_aO_bO_c$, this equals the circumradius of ABC, which is computed as

$$\frac{abc}{4(ABC)} = \frac{5\cdot 6\cdot 7}{4\sqrt{9\cdot 4\cdot 3\cdot 2}} = \boxed{\frac{35\sqrt{6}}{24}}.$$

Alternatively, consider a homothety of ratio 2 around X. Let $A'X$, $B'X$, and $C'X$ be diameters of circles O_a, O_b, and O_c, respectively. Then the homothety takes O_a to A', O_b to B', and O_c to C'; furthermore, since O_aO_b perpendicularly bisects XC, the midpoint of O_aO_b is taken to C—that is, $A'B'$ has midpoint C, and similarly $A'C'$ has midpoint B and $B'C'$ has midpoint A. Therefore, ABC is the midpoint triangle of $A'B'C'$, from which we conclude that $A'BC$, $AB'C$, and ABC' are each congruent to ABC. But r is the circumradius of each of these triangles, hence is the circumradius of ABC.

9. In quadrilateral $ABCD$, $m\angle ABD \cong m\angle BCD$ and $\angle ADB = \angle ABD + \angle BDC$. If $AB = 8$ and $AD = 5$, find BC.

Answer: $\mathbf{\frac{39}{5}}$

Solution: Note that $\angle ADB$ and $\angle CBD$ are supplementary. Therefore, we can extend AD past D to a new point C' such that $\triangle DBC \cong \triangle BDC'$ (alternatively, consider flipping $\triangle DBC$ over the altitude to $\overline{BD}$). Since $\angle ABD \cong \angle AC'B$, we have $\triangle ABD \sim \triangle AC'B$, and so

$$\frac{AC'}{AB} = \frac{AB}{AD} \implies AC' = \frac{AB^2}{AD} = \frac{64}{5}.$$

Since $AC' = AD + DC'$, we get $DC' = BC = \frac{64}{5} - 5 = \boxed{\frac{39}{5}}$.

10. A large flat plate of glass is suspended $\sqrt{2/3}$ units above a large flat plate of wood. (The glass is infinitely thin and causes no funny refractive effects.) A point source of light is suspended $\sqrt{6}$ units above the glass plate. An object rests on the glass plate of the following description. Its base is an isosceles trapezoid $ABCD$ with $AB\|DC$, $AB = AD = BC = 1$, and $DC = 2$. The point source of light is directly above the midpoint of CD. The object's upper face is a triangle EFG with $EF = 2$, $EG = FG = \sqrt{3}$. G and AB lie on opposite sides of the rectangle $EFCD$. The other sides of the object are $EA = ED = 1$, $FB = FC = 1$, and $GD = GC = 2$. Compute the area of the shadow that the object casts on the wood plate.

Answer: $\mathbf{4\sqrt{3}}$

Solution: We have $\angle A = \angle B = 120°$ and $\angle C = \angle D = 60°$ at the base, and the three "side" faces – ADE, BCF, and CDG – are all equilateral triangles. If those faces are folded down to the glass plate, they will form a large equilateral triangle of side length 3. Let E_0, F_0, and G_0 be the vertices of this equilateral triangle corresponding to E, F and G, respectively; the large triangle can be folded up along AD, CD, and BD respectively to form the three side faces of the object.

Observe that M, the midpoint of CD, is the centroid of $E_0F_0G_0$. As side ADE is folded along AD, which is perpendicular to E_0M, the projection of E onto the glass plate still lies on EM. This also holds for the projections of F and G, so projections E_1, F_1, and G_1 of E, F, and G lie on E_0M, F_0M and G_0M respectively.

Since $EFCD$ is a rectangle, E_1F_1CD is as well. Thus E_1D is perpendicular to EA. From E_0E_1 being perpendicular to AD we can conclude that E_1 should be the center of triangle ADE_0.

Symmetry gives $AE = DE = E_0E$, so AE_0DE should be a regular tetrahedron. A similar argument applies to BF_0CF.

The next step is to figure out the location of G. As $EG = \sqrt{3}$ and $DG = \sqrt{2}$, it follows that $\angle DEG$ is right. Similarly $\angle CFG$ is also right, so plane EFG should be perpendicular to plane $EFCD$.

Now we cut the whole object along the perpendicular bisector plane of AB and consider its cross-section along the plane. It will cut AB and EF along their midpoints N and P respectively. As $ABMP$ forms a regular tetrahedron of side length 1 and N is midpoint of AB, we have $NM = NP = \sqrt{3}/2$. Also $MG = \sqrt{3}$ and $\angle MPG$ is right. Let Q be the midpoint of MG; then $PQ = MQ = \sqrt{3}/2$, since right triangles are inscribed in semicircles. It follows that NPM and QMP are congruent and NP and MG are parallel. From $MG = MG_0 = \sqrt{3}$ and $NP = NM = \sqrt{3}/2$, this gives similarity between NMP and MG_0G, and $GG_0 = 2PM = 2$. Therefore $DCGG_0$ also forms a regular tetrahedron.

Since AE_0DE , BF_0CF, and CG_0DG are all regular tetrahedrons, we have three lines E_0E, F_0F, and G_0G meeting at a point X where $E_0F_0G_0X$ forms a regular tetrahedron of side length 3. Thus we finally demystified our object completely: it was obtained by cutting the regular tetrahedron $E_0F_0G_0X$ along planes EFG, ADE, BCF, CDG. Moreover we find that X is actually our point source, as it is also directly above M - both the midpoint of CD and the center of $E_0F_0G_0$ - and its height is $\sqrt{6}$, the same as that of point source. So the projection of the object to the glass plate will be exactly $E_0F_0G_0$, an equilateral triangle of side length 3. Hence the projection down to the wood plate will give an equilateral triangle of side length 4, and our answer is its area, $\boxed{4\sqrt{3}}$.