1. Compute the minimum possible value of

$$(x-1)^2+(x-2)^2+(x-3)^2+(x-4)^2+(x-5)^2$$

for real values of x.

Answer: 10

Solution: We know that this expression has to be a concave-up parabola (i.e. a parabola that faces upwards), and there is symmetry across the line $x = 3$. Hence, we conclude that the vertex of the parabola occurs at $x = 3$. Plugging in, we get $4 + 1 + 0 + 1 + 4 = \boxed{10}$.

2. Find all real values of x such that $(\frac{1}{5}(x^2-10x+26))^{x^2-6x+5} = 1$.

Answer: 1, 3, 5, 7

Solution: Clearly, the above equation holds if if $\frac{1}{5}(x^2-10x+26) = 1$ or $x^2-6x+5 = 0$, from which we obtain $3, 7$ and $1, 5$, respectively. To see that these are the only possible values for x, note that $x^2-10x+26 = (x-5)^2+1$ is always positive. Since for positive a, the function $f(y) = a^y$ is strictly increasing, the only solution to $a^y = 1$ is $y = 0$.

3. Express $\frac{2^3-1}{2^3+1} \times \frac{3^3-1}{3^3+1} \times \frac{4^3-1}{4^3+1} \times \cdots \times \frac{16^3-1}{16^3+1}$ as a fraction in lowest terms.

Answer: $\mathbf{\frac{91}{136}}$

Solution: We note

$$\prod_{n=2}^{k} \frac{n^3-1}{n^3+1} = \prod_{n=2}^{k} \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)} = \left(\prod_{n=2}^{k} \frac{n-1}{n+1}\right)\left(\prod_{n=2}^{k} \frac{n^2+n+1}{n^2-n+1}\right).$$

Each product telescopes, yielding $\frac{1\cdot 2}{k\cdot(k+1)} \cdot \frac{k^2+k+1}{3}$. Evaluating at $k = 16$ yields $\boxed{\frac{91}{136}}$.

4. If x, y, and z are integers satisfying $xyz + 4(x + y + z) = 2(xy + xz + yz) + 7$, list all possibilities for the ordered triple (x, y, z).

Answer: (1, 1, 1), (1, 3, 3), (3, 1, 3), (3, 3, 1)

Solution: Rearranging the given equality yields $xyz - 2(xy + xz + yz) + 4(x + y + z) - 8 = -1$. But the left side factors as $(x-2)(y-2)(z-2)$. Since all quantities involved are integral, we must have each factor equal to ± 1. It is easy to verify that the only possibilities for (x, y, z) are those listed.

5. The quartic (4th-degree) polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.

Answer: −24

Solution: Consider the polynomial $Q(x) = P(x) - 3$. Q has roots at $x = 2$ and $x = 3$. Moreover, since these roots are maxima, they both have multiplicity 2. Hence, Q is of the form $a(x-2)^2(x-3)^2$, and so $P(x) = a(x-2)^2(x-3)^2 + 3$. $P(1) = 0 \implies a = -\frac{3}{4}$, allowing us to compute $P(5) = -\frac{3}{4}(9)(4) + 3 = \boxed{-24}$.

6. There exist two triples of real numbers (a, b, c) such that $a - \frac{1}{b}$, $b - \frac{1}{c}$, and $c - \frac{1}{a}$ are the roots to the cubic equation $x^3 - 5x^2 - 15x + 3$ listed in increasing order. Denote those (a_1, b_1, c_1) and (a_2, b_2, c_2). If a_1, b_1, and c_1 are the roots to monic cubic polynomial f and a_2, b_2, and c_2 are the roots to monic cubic polynomial g, find $f(0)^3 + g(0)^3$.

Answer: −14

Solution: By Viéta's Formulas, we have that $f(0) = -a_1b_1c_1$ and $g(0) = -a_2b_2c_2$. Additionally, $(a-\frac{1}{b})(b-\frac{1}{c})(c-\frac{1}{a}) = -3$ and $(a-\frac{1}{b})+(b-\frac{1}{c})+(c-\frac{1}{a}) = 5$. Expanding the first expression yields $-3 = abc-\frac{1}{abc}-((a+b+c)-(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})) = abc-\frac{1}{abc}-5$. This is equivalent to $(abc)^2-2(abc)-1 = 0$, so $abc = 1 \pm \sqrt{2}$. It follows that $f(0)^3 + g(0)^3 = -(1+\sqrt{2})^3 - (1-\sqrt{2})^3 = \boxed{-14}$.

7. The function $f(x)$ is known to be of the form $\prod_{i=1}^{n} f_i(a_i x)$, where a_i is a real number and $f_i(x)$ is either $\sin(x)$ or $\cos(x)$ for $i = 1, \ldots, n$. Additionally, $f(x)$ is known to have zeros at every integer between 1 and 2012 (inclusive) except for one integer b. Find the sum of all possible values of b.

Answer: 2047

Solution: The possible values of b are the powers of two not exceeding 2012 (including $2^0 = 1$). The following proof uses the fact that the zeroes of sine and cosine are precisely numbers of the form $t\pi$ and $(t+1/2)\pi$, respectively, for t an integer.

Suppose b is not a power of 2. Then it can be written as $2^m(1+2k)$ for $m \geq 0$, $k > 0$. Since $2^m < b$, by assumption one of the f_i must have a root at 2^m. But then this same f_i must have a root at b:

- If $f_i(x) = \sin(ax)$ and $f_i(2^m) = 0$, then $2^m a = t\pi$ for some integer t, so

$$f_i(b) = \sin(ba) = \sin((1+2k)2^m a) = \sin((1+2k)t\pi) = 0.$$

- If $f_i(x) = \cos(ax)$ and $f_i(2^m) = 0$, then $2^m a = (t+1/2)\pi$ for some integer t so

$$f_i(b) = \cos(ba) = \cos((1+2k)2^m a) = \cos((1+2k)(t+1/2)\pi) = \cos((t+k+2kt+1/2)\pi) = 0$$

This is a contradiction, so b must be a power of 2.

For each b of the form 2^m, we can construct an f that works by using cosine terms to cover integers preceding b and sine terms thereafter:

$$f(x) = \left(\prod_{i=1}^{m} \cos(\pi x/2^i)\right)\left(\prod_{j=b+1}^{2012} \sin(\pi x/j)\right)$$

has a root at every positive integer at most 2012 except b.

Hence, our final answer is $1 + 2 + 4 + ... + 1024 = 2048 - 1 = \boxed{2047}$.

8. For real numbers (x, y, z) satisfying the following equations, find all possible values of $x + y + z$.

$$x^2y + y^2z + z^2x = -1$$
$$xy^2 + yz^2 + zx^2 = 5$$
$$xyz = -2$$

Answer: 2 or $\sqrt[3]{\frac{1}{2}}$

Solution 1: Let $x/y = a$, $y/z = b$, and $z/x = c$. Then $abc = 1$. By dividing the first two equations by the third equation, we have $a+b+c = -5/2$ and $1/a+1/b+1/c = ab+bc+ca = 1/2$. So a, b, c are roots of $2X^3 + 5X^2 + X - 2 = 0$. By observation, the three roots of this equation are $-2, -1$, and $1/2$. Without loss of generality, assume that $a = -1$ and $y = -x$. If $c = -2$, then we have $z = -2x$, so $xyz = 2x^3 = -2$, and thus $x = -1$. In this case we have

$$(x, y, z) = (-1, 1, 2).$$

If $c = 1/2$, then $z = x/2$, so $xyz = -x^3/2 = -2$, or $x = \sqrt[3]{4}$. In this case we have

$$(x, y, z) = \left(\sqrt[3]{4}, -\sqrt[3]{4}, \sqrt[3]{\frac{1}{2}}\right).$$

It follows that $x + y + z$ can be either $\boxed{2 \text{ or } \sqrt[3]{\frac{1}{2}}}$.

Solution 2: We have $(x+y)(y+z)(z+x) = x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2 + 2xyz = -1 + 5 + 2(-2) = 0$. Hence one of the quantities $x+y, y+z, z+x$ is 0. Suppose WLOG that $x = -y$, so that $y^2z = 2$ and $y^3 + y^2z - yz^2 = -1$. The latter equation can be rewritten as $2y/z + 2 - 2z/y = -1$, or, setting $a = y/z$, $2a + 3 - 2/a = 0$, giving the quadratic $2a^2 + 3a - 2 = (2a-1)(a+2) = 0$. Then $y = z/2$ or $y = -2z$, giving rise respectively to $z^3 = 8$ and $z^3 = 1/2$. Since $x + y = 0$, we have $x + y + z = z = \boxed{2 \text{ or } \sqrt[3]{\frac{1}{2}}}$.

9. Find the minimum value of xy, given that $x^2 + y^2 + z^2 = 7$, $xy + xz + yz = 4$, and x, y, z are real numbers.

   Answer: $\mathbf{\frac{1}{4}}$

   Solution 1: Note that $4xy = (x + y - z)^2 + 2(xy + xz + yz) - (x^2 + y^2 + z^2)$. Since $(x + y - z)^2$ is non-negative, it follows that $4xy$ is at least $2 \cdot 4 - 7 = 1$, so xy is at least $1/4$.
   We now aim to find a solution for which $xy = 1/4$ (thereby proving our lower bound to be tight). We have seen that $xy = 1/4$ implies that $x + y - z = 0$, or $z = x + y$. Substituting into $xy + xz + yz = 4$, we obtain $x^2 + 3xy + y^2 = 4$. Subtracting $xy = 1/4$ and $5xy = 5/4$ from this equation yields $(x + y)^2 = 15/4$ and $(x - y)^2 = 11/4$. Thus, one solution is $x + y = \frac{\sqrt{15}}{2}, x - y = \frac{\sqrt{11}}{2}$, or $x = \frac{\sqrt{15}+\sqrt{11}}{4}, y = \frac{\sqrt{15}-\sqrt{11}}{4}, z = \frac{\sqrt{15}}{2}$.

   Solution 2: We have $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = 7 + 2 \cdot 4 = 15$. Note that minimizing xy is equivalent to maximizing $xz + yz = (x + y)z$. Also note that this maximum value of $(x + y)z$ is certainly positive, and in fact we can assume $x + y$ and z are both positive, since otherwise we can replace x, y, z with $-x, -y, -z$ without changing any of our quantities. We can then apply AM-GM to find that $\sqrt{(x + y)z} \le \frac{x+y+z}{2}$, or $(x + y)z \le \frac{(x+y+z)^2}{4} = \frac{15}{4}$. Hence $xy \ge 4 - \frac{15}{4} = \boxed{\frac{1}{4}}$ and we proceed as in the first solution.

10. Let $X_1, X_2, \ldots, X_{2012}$ be chosen independently and uniformly at random from the interval $(0, 1]$. In other words, for each X_n, the probability that it is in the interval $(a, b]$ is $b - a$. Compute the probability that $\lceil \log_2 X_1 \rceil + \lceil \log_4 X_2 \rceil + \cdots + \lceil \log_{4024} X_{2012} \rceil$ is even. (Note: For any real number a, $\lceil a \rceil$ is defined as the smallest integer not less than a.)

    Answer: $\mathbf{\frac{2013}{4025}}$

    Solution: To simplify notation, define $Y_n = \lceil \log_{2n} X_n \rceil$.

    We begin by computing the probability that Y_n is odd. $Y_n = -1$ if $-2 < \log_{2n} X_n \le -1$, or $\frac{1}{(2n)^2} < X_n \le \frac{1}{2n}$. Similarly, $Y_n = -3$ if $\frac{1}{(2n)^4} < X_n \le \frac{1}{(2n)^3}$, and so on. Adding up the lengths of these intervals, we see that the probability that Y_n is odd is $\sum_{k=1}^{\infty} \frac{1}{(2n)^{2k-1}} - \frac{1}{(2n)^{2k}}$. This is a geometric series with first term $\frac{1}{2n}(1-\frac{1}{2n})$ and ratio $\frac{1}{(2n)^2}$, so its sum is

$$\frac{\frac{1}{2n}(1-\frac{1}{2n})}{1-\frac{1}{(2n)^2}} = \frac{\frac{1}{2n}}{(1+\frac{1}{2n})} = \frac{1}{2n+1}.$$

Armed with this fact, we are now ready to solve the problem. One way to continue would be to note that the probability that Y_1 is even is $2/3$, the probability that Y_1+Y_2 is even is $3/5$, the probability that $Y_1+Y_2+Y_3$ is even is $4/7$ and to show by induction that the probability that $Y_1+\cdots+Y_n$ is even is $\frac{n+1}{2n+1}$. Below, we present an alternate approach.

Note that $Y_1+Y_2+\cdots+Y_{2012}$ is even if and only if $(-1)^{Y_1+Y_2+\cdots+Y_{2012}} = 1$. Rewrite $(-1)^{Y_1+Y_2+\cdots+Y_{2012}}$ as $(-1)^{Y_1}(-1)^{Y_2}\cdots(-1)^{Y_{2012}}$, and note that because the Y_n are independent,

$$E\left[(-1)^{Y_1}(-1)^{Y_2}\cdots(-1)^{Y_{2012}}\right] = E[(-1)^{Y_1}]E[(-1)^{Y_2}]\cdots E[(-1)^{Y_{2012}}], \tag{1}$$

where E denotes the expected value of the quantity. But $E[Y_n] = (+1)\cdot P(Y_n \text{ is even}) + (-1)\cdot P(Y_n \text{ is odd})$. We computed earlier that the probability that Y_n is odd is $\frac{1}{2n+1}$, so $E[Y_n] = \frac{2n-1}{2n+1}$ and product in (1) is $\frac{1}{3}\cdot\frac{3}{5}\cdots\frac{4023}{4025}$, which telescopes to yield $\frac{1}{4025}$. Let p be the probability that $Y_1+Y_2+\cdots+Y_{2012}$ is even. We just found that $(+1)(p)+(-1)(1-p) = \frac{1}{4025}$, which we can solve to obtain $p = \boxed{\frac{2013}{4025}}$.