1. Define a number to be *boring* if all the digits of the number are the same. How many positive integers less than 10000 are both prime and boring?

   Answer: 5

   Solution: The one-digit boring primes are 2, 3, 5, and 7. The only two-digit boring prime is 11, since 11 divides all other two-digit boring numbers. No three-digit boring numbers are prime, since 111 divides all of them and $111 = 3 \times 37$. No four-digit boring numbers are prime since they are all divisible by 11. Therefore, there are $\boxed{5}$ positive integers less than 10000 which are both prime and boring.

2. Find the sum of all integers x, $x \geq 3$, such that 201020112012_x (that is, 201020112012 interpreted as a base x number) is divisible by $x - 1$.

   Answer: 32

   Solution: Note that $x \equiv 1 \pmod{x-1}$, and so $x^n \equiv 1 \pmod{x-1}$ for all positive integers n. Hence, the number 201020112012 in base x is congruent to the sum of its digits $= 12 \pmod{x-1}$. Therefore, we simply need to find all $x \geq 3$ such that $12 \equiv 0 \pmod{x-1} \iff (x-1) \mid 12$, so $x - 1 = 1, 2, 3, 4, 6, 12 \implies x = 3, 4, 5, 7, 13$ (since $x \geq 3$). Hence, our answer is $\boxed{32}$.

3. Given that $\log_{10} 2 \approx 0.30103$, find the smallest positive integer n such that the decimal representation of 2^{10n} does not begin with the digit 1.

   Answer: 30

   Solution: Observe that 2^{10n} begins with the digit 1 when the fractional part of $\log_{10} 2^{10n} = 10n \log_{10} 2 \approx 3.0103n$ is $< \log_{10} 2$. Therefore, we want $0.0103n > \log_{10} 2 \approx 0.30103 \Rightarrow n \geq \boxed{30}$.

4. Two different squares are randomly chosen from an 8×8 chessboard. What is the probability that two queens placed on the two squares can attack each other? Recall that queens in chess can attack any square in a straight line vertically, horizontally, or diagonally from their current position.

   Answer: $\frac{13}{36}$

   Solution: All squares that are on the edge of the chessboard can hit 21 squares; there are 28 such squares. Now consider the 6×6 chessboard that is obtained by removing these bordering squares. The squares on the edge of this board can hit 23 squares; there are 20 of these squares. Now we consider the 12 squares on the boundary of the 4×4 chessboard left; each of these squares can hit 25 squares. The remaining 4 can hit 27 squares. The probability then follows as $\dfrac{21 \times 28 + 23 \times 20 + 25 \times 12 + 27 \times 4}{64 \times 63} = \boxed{\dfrac{13}{36}}$.

5. A short rectangular table has four legs, each 8 inches long. For each leg Bill picks a random integer x, $0 \leq x < 8$ and cuts x inches off the bottom of that leg. After he's cut all four legs, compute the probability that the table won't wobble (i.e. that the ends of the legs are coplanar).

   Answer: $\frac{43}{512}$

   Solution: We can describe a table by a, b, c, d ($1 \leq a, b, c, d \leq 8$), giving the final lengths of each of the four legs in clockwise order. How much a table is tipped north to south will be determined by the difference between the lengths a, b and c, d, and east to west by the difference between the lengths a, c and b, d. Hence, for the table to not wobble we must have $a - c = b - d \iff a - b = c - d \iff a + d = b + c$.

We can therefore split into cases based on $S = a + d = b + c$. The number of ordered pairs (x, y) such that $x + y = S$ and $1 \leq x, y \leq 8$ is $T_S = 8 - |S - 9|$ (similar to adding the values on two 8-sided dice). The number of choices for (a, d) is therefore T_S and the number of choices for (b, c) is T_S, so the number of choices for (a, b, c, d) is T_S^2.

Summing over all possible values of S this is

$$\begin{aligned}
T_2^2 + \ldots T_{16}^2 &= (8 - |2 - 9|)^2 + \cdots + (8 - |16 - 9|)^2 \\
&= 1^2 + 2^2 + \cdots + 7^2 + 8^2 + 7^2 + \cdots + 2^2 + 1^2 \\
&= 2\left(1^2 + \cdots + 7^2\right) + 8^2 \\
&= 2 \cdot \frac{7 \cdot 8 \cdot 15}{6} + 8^2 \\
&= 7 \cdot 8 \cdot 5 + 8^2 \\
&= 8\,(7 \cdot 5 + 8)\,.
\end{aligned}$$

Hence, the probability is

$$\frac{8\,(7 \cdot 5 + 8)}{8^4} = \frac{7 \cdot 5 + 8}{8^3} = \boxed{\frac{43}{512}}.$$

6. Two ants are on opposite vertices of a regular octahedron (an 8-sized polyhedron with 6 vertices, each of which is adjacent to 4 others), and make moves simultaneously and continuously until they meet. At every move, each ant randomly chooses one of the four adjacent vertices to move to. Eventually, they will meet either at a vertex (that is, at the completion of a move) or on an edge (that is, in the middle of a move). Find the probability that they meet on an edge.

Answer: $\mathbf{\frac{2}{11}}$

Solution: If the two ants are not on the same vertex, they can either be on opposite vertices or on adjacent vertices. Let x and y be the probabilities that the ants will eventually meet on an edge when starting out from opposite vertices and from adjacent vertices, respectively. From opposite vertices, one of the ants must move to one of the remaining four vertices, which are all equivalent with respect to the other ant. That ant can either meet the first ant at a vertex, become adjacent to it (two ways to do this), or again become opposite from it. So

$$x = \frac{1}{4}x + \frac{1}{2}y.$$

If the two ants are adjacent, the cases become slightly more complicated. If the first ant moves towards the second ant, the second ant can move towards it (meeting on an edge); otherwise they will be adjacent. If the first ant moves away from the second ant, they will become adjacent no matter what the second ant does. If the first ant moves to the side (two ways to do this), they will be opposite if the second ant chooses the other direction, and will meet at a vertex if it chooses the same direction. Otherwise they will be adjacent. So

$$y = \frac{1}{8}x + \frac{11}{16}y + \frac{1}{16}.$$

This system of equations is easily solved to obtain $x = \boxed{\frac{2}{11}}$.

7. Determine the greatest common divisor of the elements of the set $\{n^{13} - n \mid n \text{ is an integer}\}$.

Answer: 2730

Solution: Let D be the desired greatest common divisor. By Fermat's Little Theorem we have: $n^{13} \equiv (n^6)^2(n) \equiv (n^3)^2(n) \equiv n^4 \equiv n^2 \equiv n \mod 2$.

Hence $2 \mid (n^{13} - n)$ for all n, so $2 \mid D$. Similarly we can show that $p \mid D$ for $p \in \{3, 5, 7, 13\}$. Since these are all prime, their product, 2730, divides D.

$2^{13} - 2 = 8190 = 3(2730)$, so D is either 2730 or 3(2730). As $3^{13} - 3 = 3(3^{12} - 1)$ is not divisible by 9, $D = \boxed{2730}$.

8. We say that a set of positive integers S, all greater than 1, *covers* an integer x if for every pair of integers k and l such that $2 \le k < l \le x$, we have $\left\lceil \frac{k}{y} \right\rceil \neq \left\lceil \frac{l}{y} \right\rceil$ for at least one integer y in S. How many numbers are in the smallest set S which covers 30?

Answer: 10

Solution: We claim that the set of all primes less than 30 is the smallest set S which covers 30. We first prove that no smaller set can exist; assume that one does exist. This smaller set cannot contain some prime $p < 30$. Note that, therefore, p and $p+1$ are indistinguishable. This is a contradiction. We must now show that the set of all primes less than 30 is valid. Since two consecutive integers are relatively prime, if some prime p divides k, it does not divide $k + 1$. Therefore, $\left\lceil \frac{k+1}{p} \right\rceil = \frac{k}{p} + 1$. Therefore, the set of all primes less than k will always cover k. It remains to compute the number of primes less than 30. There are $\boxed{10}$ primes less than 30.

9. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3}$. You can use Euler's result $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

Answer: $10 - \pi^2$

Solution: We use partial fractions repeatedly to obtain that

$$\begin{aligned}
\frac{1}{n^3(n+1)^3} &= \left(\frac{1}{n} - \frac{1}{n+1}\right)^3 \\
&= \frac{1}{n^3} - \frac{1}{(n+1)^3} - 3\frac{1}{n(n+1)}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{n^3} - \frac{1}{(n+1)^3} - 3\left(\frac{1}{n} - \frac{1}{n+1}\right)^2 \\
&= \frac{1}{n^3} - \frac{1}{(n+1)^3} - \frac{3}{n^2} - \frac{3}{(n+1)^2} + \frac{6}{n(n+1)} \\
&= \frac{1}{n^3} - \frac{1}{(n+1)^3} - \frac{3}{n^2} - \frac{3}{(n+1)^2} + \frac{6}{n} - \frac{6}{n+1}.
\end{aligned}$$

Then by taking sums and using the property of telescoping sums we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3} = 1 - 3\frac{\pi^2}{6} - 3\left(\frac{\pi^2}{6} - 1\right) + 6 = \boxed{10 - \pi^2}.$$

10. We say that two polynomials $F(x)$ and $G(x)$ are equivalent mod 5 if and only if $F(x) - G(x) = 5 \cdot H(x)$ for some integer polynomial $H(x)$. We say that $F(x)$ has n as a root mod 5 if and only

if $5 \mid F(n)$. How many inequivalent integer polynomials mod 5 of degree at most 3 do not have any integer roots mod 5?

Answer: 204

Solution: Observe that a polynomial

$$I_a(X) = 1 - (X - a)^{p-1}$$

takes value 1 at a and 0 elsewhere in mod p, by Fermat's little theorem. Thus for any polynomial F mod p, we have

$$F(n) = \sum_{a=0}^{p-1} F(a) I_a(n) \pmod{p}$$

for all n. Now the polynomial of degree $\leq p-1$

$$F(X) - \sum_{a=0}^{p-1} F(a) I_a(X)$$

has $0, 1, \cdots, (p-1)$ as roots, thus it should be zero mod p. This means that polynomials mod p of degree less than p have one-to-one correspondence to p-tuples of $(F(0), F(1), \cdots, F(p-1))$ mod p. Since F not having any roots is equivalent to that none of $F(a)$ is zero, there are $(p-1)^p$ ways to choose $(F(0), F(1), \cdots, F(p-1))$. This gives the answer to the first part.

For the second part, note that coefficient of X^{p-1} in $\sum_{a=0}^{p-1} F(n) I_a(X)$ is $-\sum F(a)$, so it is equivalent to find number of p-tuples $(F(0), F(1), \cdots, F(p-1))$ satisfying $F(a) \neq 0 \pmod{p}$ for all a and $\sum F(a) = 0 \pmod{p}$. We define

$$\begin{aligned} A_n = &\text{ the number of } n \text{ tuples } (a_1, \cdots, a_n) \text{ satisfying} \\ &1 \leq a_i \leq p-1, \quad p \mid a_1 + \cdots + a_n \end{aligned}$$

and the problem is to find A_p. We establish the recurrence relation on A_n. For the initial condition we have $A_1 = 0$ and $A_2 = p - 1$. For $n > 2$, note that if $(a_1, a_2, \cdots, a_n)$ is counted in A_n, then a_n is uniquely chosen to be $\equiv -(a_1 + \cdots + a_{n-1}) \pmod{p}$ unless $a_1 + \cdots + a_{n-1}$ is not divisible by p. This is equivalent to say that A_n is same as the number of $(n-1)$-tuples with their sum not divisible by p. This gives the recurrence

$$A_n = (p-1)^{n-1} - A_{n-1}$$

and by solving it we have

$$A_n = (p-1)^{n-1} - (p-1)^{n-2} + (p-1)^{n-3} - \cdots + (-1)^{n-2}(p-1).$$

So the answer is $A_p = \frac{(p-1)^{np} + (-1)^{p-1}(p-1)}{(p-1)+1} = \frac{(p-1)^p - (p-1)}{p}$. Evaluating at $p = 5$, we get $\frac{4^5 - 4}{5} = \frac{1020}{5} = \boxed{204}$.