2011 SMT POWER ROUND: POLES AND POLARS

Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve a multi-part, proof-oriented question. Answers are to be submitted on clearly labeled sheets of scratch paper. Every problem must be written *on its own sheet.* Every submitted page should contain the *team number, problem number, and a page number.*
No calculators.

1. Definition and Basic Properties

In the Cartesian plane, the *polar* of the point $P = (p_x, p_y)$ with respect to the unit circle is the line $p_x x + p_y y = 1$. The *pole* of the line $l : l_x x + l_y y = 1$ with respect to the unit circle is the point (l_x, l_y).

1. Graph the polars (or the poles) of the following figures, together with the original figures (separate graph for each part). Also draw your coordinate axes. You don't need to justify. [2 pts each]
   (1) $A(1, 1)$ (the "A" is just a label for the point $(1, 1)$)
   (2) $b : y = x + 2$
   (3) $C(-0.6, 0.8)$
   (4) $d : x = 2y + 2$
   (5) E : the pole of line a, the polar of A
   (6) f : the polar of the intersection of lines a and b
2. How many intersections are there between the polar of P and the unit circle, given that P is (1) inside, (2) on, and (3) outside the unit circle? Justify. [12 pts]

The *reciprocation* transform takes a point which is not the origin to its polar, and a line which do not go through the origin to its pole. It is easy to see that reciprocation establishes a bijection between non-origin points and lines which do not go through the origin, and it is its own inverse: l is the polar of the point P if and only if P is the pole of the line l.

3. The more common definitions of poles and polars are the following:
   Define the *inversion* of the nonorigin point P with respect to the unit circle as follows: it is the point P' on the ray $\overline{OP}$ satisfying $OP \cdot OP' = 1$. Then define the polar of P to be the line going through P' which is perpendicular to OP. Also for a line l which does not go through the origin, define the pole of l to be the inversion of the foot of the perpendicular from O to l.
   Show that this definition is equivalent to our initial definition. [12 pts]
   (For future reference, if you take the inversion of all the points on a circle that goes through the origin, you get a line. If you take the inversion of all the points on a circle that does NOT go through the origin, you get another circle.)
4. Generalize the concept of reciprocation to reciprocation around any circle $(x - a)^2 + (y - b)^2 = r^2$ (think of the reciprocation we have defined as reciprocation around the unit circle). Find the equation of the polar of (p, q) (where $(p, q) \neq (a, b)$) and the coordinates of the pole of $c(x - a) + d(y - b) = 1$ (where $(c, d) \neq (0, 0)$). Your definition should be consistent with the rescaling and translation, but you don't need to justify your answer. [12 pts]

2. The Duality Principle

For the remaining problems you don't need to use generalized reciprocation unless noted.

5. (1) Prove that A is on the polar of B if and only if B is on the polar of A using the definition given in section 1. (2) Prove the same thing using the definition given in problem 3. [12 pts]
6. The following are corollaries of 5. Let A, B, C be non-origin points, and a, b, c be their respective polars. Prove the following: [7 pts]
   (a) The point A is on b if and only if the point B is on a.
   (b) The pole of AB is the intersection of a and b. Conversely, the polar of the intersection of a and b is AB.

Date: February 19, 2011.

(c) Points A, B, C are collinear (on the same line) if and only if lines a, b, c are concurrent (go through the same point).

In geometry, *duality* refers to geometric transformations that replace points by lines and lines by points while preserving *incidence properties*, the relation of a line going through a point. It is easy to see that reciprocation is a duality for nonorigin points and lines not going through the origin. This leads to a general principle called the *duality principle*: any theorem about incidences between points and lines may be transformed into another theorem about lines and points, by a substitution of the appropriate words. The transformed theorem is sometimes called the *dual theorem* or *reciprocal*.

7. State the dual theorem of the following. [6 pts]
   (Pappus's theorem) Given A, B, C collinear and D, E, F collinear (not necessarily in that order), the three intersection points $X = BF \cap CE$, $Y = AF \cap CD$, and $Z = AE \cap BD$ are also collinear.
8. Explain how to generalize the duality principle to theorems which also include incidences between a given circle and some points. Use this generalization to state the dual theorem of the following. Your generalized reciprocation from problem 4 will be useful. [10 pts]
   (Pascal's theorem) For a cyclic hexagon $ABCDEF$ (not necessarily around the circle in that order), the three intersections of opposite sides $X = AB \cap DE$, $Y = BC \cap EF$, $Z = CD \cap FA$ are collinear.

3. Reciprocation and Cyclic Quadrilaterals

9. Given a circle, let P be a point inside the circle, and AC and BD be two chords of the circle which go through P. Let Q be the intersection of the lines AD and BC (outside the circle). Then Q is on the polar of P. Prove this statement following the steps below: [21 pts]
   (1) Let X be the point on the segment AD satisfying $AX/XD = AQ/QD$. Supposing that the circle is the unit circle, points A and D have coordinates $A = (x_a, y_a)$ and $D = (x_d, y_d)$, and the ratio $AQ : QD = m : n$ (where $m, n \neq 0$), express the coordinates of Q and X in terms of x_a, y_a, x_d, y_d, m, n. We say that X is the *harmonic conjugate* of the three points A, D, Q.
   (2) Show that X is on the polar of Q.
   (3) Define Y be the harmonic conjugate of B, C, Q. Show that AC, XY, BD intersect at one point. To show this, let P_1 and P_2 be the intersection of AC and XY, BD and XY respectively, and calculate the ratio P_1X/P_1Y and P_2X/P_2Y using Menelaus' theorem (if ABC is a triangle and P, Q, R are points on the lines AB, BC, and CA respectively, then P, Q, R are collinear if and only if $\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1$) to see that those two coincide.
   (4) Conclude that P is on the polar of Q.
10. For cyclic quadrilateral $ABCD$, we can consider three points $P = AC \cap BD$, $Q = AD \cap BC$ and $R = AB \cap CD$. Show that the polar of each point goes through other two ponts. What can be said about the orthocenter of PQR? [12 pts]
11. Prove the following "six points on the line" property: for cyclic quadrilateral $ABCD$ we have (1) the intersection of AC and BD, (2) the intersection of AD and BC, (3) the intersection of the tangent at A and the tangent at B, (4) the intersection of the tangent at C and the tangent at D, and (5) and (6) the tangent points of the two tangents from $AB \cap CD$ to the circle. The property is that all of these points are on the same line. [12 pts]
12. This is not related to poles and polars, but can you prove that (7) the intersection of the circumcircles of ADP and BCP (other than P) is also on the line? [12 pts]

4. Conic Sections

13. Take a circle centered at A with radius r, a point O inside the circle, and a circle centered at O with radius 1. The poles of tangents to circle A with respect to circle O form a conic section. (1) Is it an ellipse, parabola, or hyperbola? (2) What is its relationship to the polars of the points on circle A with respect to circle O? What are the conic sections we get if O is (3) on circle A, or (4) outside? You should justify each of your choices under the assumption that these loci are indeed conic sections, but you do not need to check their precise shape. [12 pts]
14. Take circles centered at A and O as in problem 13, with O lying on circle A. Take the associated conic section, again as in problem 13. Consider the feet of the perpendiculars from O to the tangent lines of this conic section; find and justify the locus of these feet. [12 pts]

15. Again set your diagram up as in problem 13 (this time, make no assumptions about the placement of O relative to circle A). Suppose point P lies on the associated conic section (i.e. it is the pole of some tangent to circle A with respect to circle O). Let a be the polar of A with respect to circle O. Prove that the distance OP is equal to ϵ times the distance from P to a, where ϵ is a constant that does not depend on P. (Hint: it may help to consider the point M, defined as the intersection between OA and the polar of P.) [14 pts]

5. Counting

Taking the dual of certain counting problems can make the counting easier. Apply the idea of duality to help solve the following problem.

16. For any $n > 0$, take n distinguishable points $P_1, ..., P_n$ in the plane. For any line ℓ not going through any of the points, let the "linear partition of $P_1, ..., P_n$ by ℓ", which we denote $M_\ell(P_1, ..., P_n)$, be the unordered pair $\{L, R\}$ where L is the set of points on one side of the line and R is the set of points on the other side of the line. Let $C(P_1, ..., P_n)$ be the number of distinct unordered pairs $M_\ell(P_1, ..., P_n)$ as ℓ ranges over all lines. i.e., $C(P_1, ..., P_n)$ is the number of different ways we can split $P_1, ..., P_n$ with lines.

    Here are some examples. For one point, $C = 1$ because there is only one way to "split" the set of one point. For two points, $C = 2$ because we can have both points on one side of the line or each point on an opposite side of the line. For three collinear points P_1, P_2, P_3, $C = 3$ because the linear partitions are $\mid P_1, P_2, P_3$ and $P_1 \mid P_2, P_3$ and $P_1, P_2 \mid P_3$. For three points on the vertices of an equilateral triangle, $C = 4$ because we can have all the points on the same side of the line or we can split off any one of the three points from the other two.

    What is the maximum of $C(P_1, ..., P_n)$ over all choices of n points? Prove that your answer is the maximum. Duality will be helpful. [22 pts]