1. Triangle ABC has side lengths $BC = 3$, $AC = 4$, $AB = 5$. Let P be a point inside or on triangle ABC and let the lengths of the perpendiculars from P to BC, AC, AB be D_a, D_b, D_c respectively. Compute the minimum of $D_a + D_b + D_c$.

   Answer: $\mathbf{\frac{12}{5}}$

   Let $M = D_a + D_b + D_c$. Notice that the total area of ABC is $(1/2)(3D_a + 4D_b + 5D_c) = (1/2)[5(D_a + D_b + D_c) - 2D_a - D_b] = (1/2)(5M - 2D_a - D_b)$. Of course, since ABC is a right triangle, we know its area is $3 \cdot 4/2 = 6$. We thus have $M = (1/5)(2 \cdot 6 + 2D_a + D_b)$. This is clearly minimized when $D_a = D_b = 0$, that is, when we set P to coincide with C. We then have $M = 12/5$.

   Note: it is also possible to conclude this from a few applications of the triangle inequality.

2. Pentagon $ABCDE$ is inscribed in a circle of radius 1. If $\angle DEA \cong \angle EAB \cong \angle ABC$, $m\angle CAD = 60°$, and $BC = 2DE$, compute the area of $ABCDE$.

   Answer: $\mathbf{\frac{33\sqrt{3}}{28}}$

   Looking at cyclic quadrilaterals $ABCD$ and $ACDF$ tells us that $m\angle ACD = m\angle ADC$, so $\triangle ACD$ is equilateral and $m\angle DEA = 120°$. Now, if we let $m\angle EAD = \theta$, we see that $m\angle CAB = 60° - \theta \implies m\angle ACB = \theta \implies \triangle AED \cong \triangle CBA$. Now all we have to do is calculate side lengths. After creating some $30° - 60° - 90°$ triangles, it becomes evident that $AC = \sqrt{3}$. Now let $AB = x$, so $BC = 2x$. By applying the Law of Cosines to triangle ABC, we find that $x^2 = \frac{3}{7}$. Hence, the desired area $(ABCDE) = (ACD) + 2(ABC) = \frac{(\sqrt{3})^2\sqrt{3}}{4} + 2 \cdot \frac{1}{2}(x)(2x)(\sin 120°) = \frac{33\sqrt{3}}{28}$.

3. Let circle O have radius 5 with diameter $\overline{AE}$. Point F is outside circle O such that lines $\overline{FA}$ and $\overline{FE}$ intersect circle O at points B and D, respectively. If $FA = 10$ and $m\angle FAE = 30°$, then the perimeter of quadrilateral $ABDE$ can be expressed as $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are rational. Find $a + b + c + d$.

   Answer: 15

   After some angle chasing, we find that $m\angle DBF = m\angle DFB = 75°$, which implies that $DF = DB$. Hence the desired perimeter is equal to $AF - BF + AE + FE = 20 - BF + FE$.

   By the law of sines, $\frac{FE}{\sin 30°} = \frac{10}{\sin 75°} \implies FE = \frac{5}{\frac{\sqrt{6}+\sqrt{2}}{4}} = 5\sqrt{6} - 5\sqrt{2}$.

   Now, to find $\overline{BF}$, draw the altitude from O to $\overline{AB}$ intersecting $\overline{AB}$ at P. This forms a $30° - 60° - 90°$ triangle, so we can see that $AP = 5\sqrt{3}/2 = \frac{10-BF}{2} \implies BF = 10 - 5\sqrt{3}$. Hence, the desired perimeter is $20 + (5\sqrt{6} - 5\sqrt{2}) - (10 - 5\sqrt{3}) = 10 - 5\sqrt{2} + 5\sqrt{3} + 5\sqrt{6}$, so the answer is $10 - 5 + 5 + 5 = 15$.

4. Let ABC be any triangle, and D, E, F be points on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $CD = 2BD$, $AE = 2CE$ and $BF = 2AF$. $\overline{AD}$ and $\overline{BE}$ intersect at X, $\overline{BE}$ and $\overline{CF}$ intersect at Y, and $\overline{CF}$ and $\overline{AD}$ intersect at Z. Find $\frac{Area(\triangle ABC)}{Area(\triangle XYZ)}$.

   Answer: 7

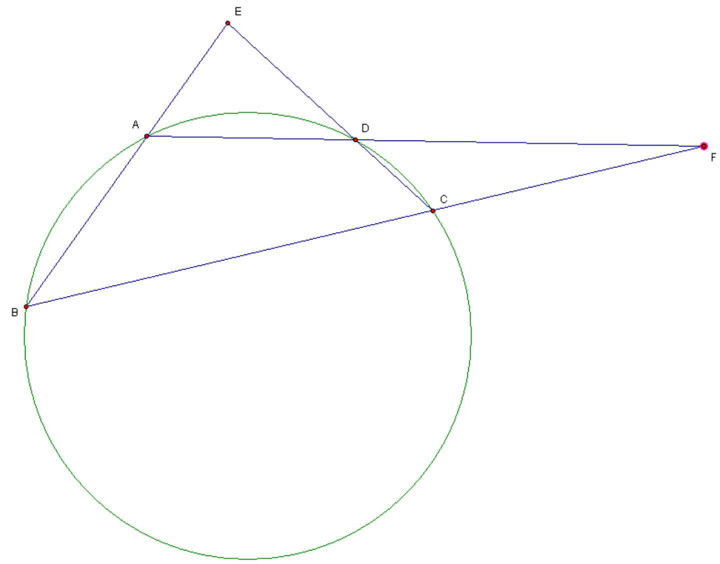
   Using Menelaus's Theorem on $\triangle ABD$ with collinear points F, X, C and the provided ratios gives $DX/XA = 4/3$. Using Menelaus's Theorem on $\triangle ADC$ with collinear points B, Y, E gives $AY/YD = 6$. We conclude that AX, XY, YD are in length ratio $3 : 3 : 1$. By symmetry, this also applies to the segments CZ, ZX, XF and BY, YZ, ZE. Repeatedly using the fact that the area ratio of two triangles of equal height is the ratio of their bases, we find $[ABC] = (3/2)[ADC] = (3/2)(7/3)[XYC] = (3/2)(7/3)(2)[XYZ] = 7[XYZ]$, or $[ABC]/[XYZ] = 7$.

   Alternate Solution

   Stretching the triangle will preserve ratios between lengths and ratios between areas, so we may assume that $\triangle ABC$ is equilateral with side length 3. We now use mass points to find the length of XY. Assign a mass of 1 to A. In order to have X be the fulcrum of $\triangle ABC$, C have mass 2 and B must have mass 4. Hence, $BX : XE = 4 : 3$ and $AX : XD = 6 : 1$, the latter of which also equals $BY : YE$ by symmetry. Hence, $XY = \frac{3}{7}BE$. To find BE, we apply the Law of Cosines to $\triangle CBE$ to get that $BE^2 = 1^2 + 3^2 - 2 \cdot 1 \cdot 3 \cdot \cos 60° = 7 \implies XY = \frac{3\sqrt{7}}{7}$. Since $\triangle XYZ$ must be equilateral by symmetry, the desired ratio equals $(\frac{AB}{XY})^2 = 7$.

5. Let $ABCD$ be a cyclic quadrilateral with $AB = 6$, $BC = 12$, $CD = 3$, and $DA = 6$. Let E, F be the intersection of lines AB and CD, lines AD and BC respectively. Find EF.

   Answer: $\mathbf{10\sqrt{2}}$



We have $\triangle ADE \sim \triangle CBE$, and their length ratio is $AD : CB = 1 : 2$. Let $AE = p$ and $DE = q$. Then we have $AB = BE - AE = 2DE - AE = 2q - p$ and $CD = 2p - q$. Solving for p and q, we have $p = 4$ and $q = 5$. Similarly we have $FC = 8$ and $FD = 10$. Let $\angle B = \theta$. Then $\angle FDE = \pi - \theta$. Apply the Law of Cosines to $\triangle EBF$ to get

$$EF^2 = BE^2 + BF^2 - 2BE \cdot BF \cdot \cos\theta = 10^2 + 20^2 - 2 \cdot 10 \cdot 20 \cos\theta = 500 - 400\cos\theta$$

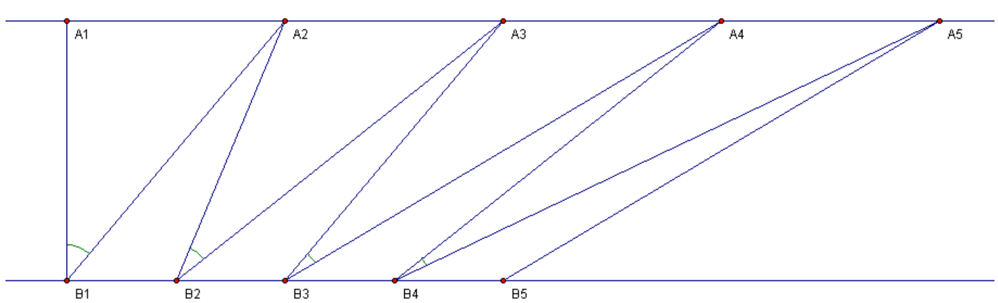
and to $\triangle EDF$ to get

$$EF^2 = DE^2 + DF^2 + 2 \cdot DE \cdot DF \cos\theta = 5^2 + 10^2 - 2 \cdot 5 \cdot 10 \cos\theta = 125 + 100\cos\theta.$$

Solving for EF^2, we get $EF^2 = 200$.

6. Two parallel lines l_1 and l_2 lie on a plane, distance d apart. On l_1 there are an infinite number of points $A_1, A_2, A_3, \cdots$, in that order, with $A_nA_{n+1} = 2$ for all n. On l_2 there are an infinite number of points $B_1, B_2, B_3, \cdots$, in that order and in the same direction, satisfying $B_nB_{n+1} = 1$ for all n. Given that A_1B_1 is perpendicular to both l_1 and l_2, express the sum $\sum_{i=1}^{\infty} \angle A_iB_iA_{i+1}$ in terms of d.

   Answer: $\boldsymbol{\pi - \tan^{-1}(\frac{1}{d})}$ **(or** $\boldsymbol{\pi/2 + \tan^{-1} d}$ **or other equivalent form)**



Construct points $C_1, C_2, C_3, \cdots$ on l_1 progressing in the same direction as the A_i such that $C_1 = A_1$ and $C_nC_{n+1} = 1$. Thus we have $C_1 = A_1$, $C_3 = A_2$, $C_5 = A_3$, etc., with $C_{2n-1} = A_n$ in general. We can write $\angle A_iB_iA_{i+1} = \angle C_{2i-1}B_iC_{2i+1} = \angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1}$. Observe that $\triangle C_iB_iC_k$ (for any k) is a right triangle with legs of length d and $k-i$, and $\angle C_iB_iC_k = \tan^{-1}\frac{k-i}{d}$. So $\angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1} = \tan^{-1}\frac{i+1}{d} - \tan^{-1}\frac{i-1}{d}$. The whole sum is therefore

$$\sum_{i=1}^{\infty}\left(\tan^{-1}\frac{i+1}{d} - \tan^{-1}\frac{i-1}{d}\right)$$

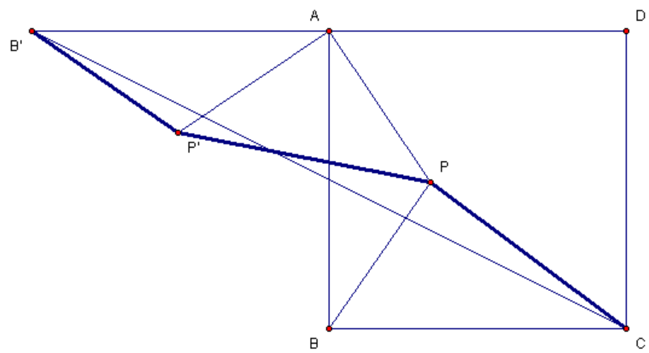
which has nth partial sum

$$\tan^{-1}\frac{n+1}{d} + \tan^{-1}\frac{n}{d} - \tan^{-1}\frac{1}{d}$$

so it converges to $\boxed{\pi - \tan^{-1}\frac{1}{d}}$.

7. In a unit square $ABCD$, find the minimum of $\sqrt{2}AP + BP + CP$ where P is a point inside $ABCD$.

Answer: $\mathbf{\sqrt{5}}$



Rotate triangle APB around A by 90 degrees as in the given figure. Let P' and B' be the rotated images of P and B respectively. Then we have $B'P' = BP$, $P'P = \sqrt{2}AP$ so

$$\sqrt{2}AP + BP + CP = CP = PP' + P'B' \leq CB' = \sqrt{5}.$$

8. We have a unit cube $ABCDEFGH$ where $ABCD$ is the top side and $EFGH$ is the bottom side with E below A, F below B, and so on. Equilateral triangle BDG cuts out a circle from the cube's inscribed sphere. Find the area of the circle.

Answer: $\boldsymbol{\frac{\pi}{6}}$

Consider the cube to be of side length 2 and divide the answer by 4 later. Set the coordinates of the vertices of the cube to be $(\pm 1, \pm 1, \pm 1)$. Then the plane going through an equilateral triangle can be described by the equation $x+y+z = 1$. The distance to the plane from the origin is $\frac{1}{\sqrt{3}}$, as $(1/3, 1/3, 1/3)$ is the foot of the perpendicular from $(0, 0, 0)$. Thus the radius of the circle is $\sqrt{1 - (\frac{1}{\sqrt{3}})^2} = \sqrt{\frac{2}{3}}$, so the area is $\frac{2}{3}\pi$. In the case of the unit cube we should divide this by 4 to get the answer $\frac{\pi}{6}$.

9. We have a circle O with radius 10 and four smaller circles O_1, O_2, O_3, O_4 of radius 1 which are internally tangent to O, with their tangent points to O in counterclockwise order. The small circles do not intersect each other. Among the two common external tangents of O_1 and O_2, let l_{12} be the one which separates O_1 and O_2 from the other two circles, and let the intersections of l_{12} and O be A_1 and B_2, with A_1 denoting the point closer to O_1. Define l_{23}, l_{34}, l_{41} and $A_2, A_3, A_4, B_3, B_4, B_1$ similarly. Suppose that the arcs A_1B_1, A_2B_2, and A_3B_3 have length π, $3\pi/2$, and $5\pi/2$ respectively. Find the arc length of A_4B_4.

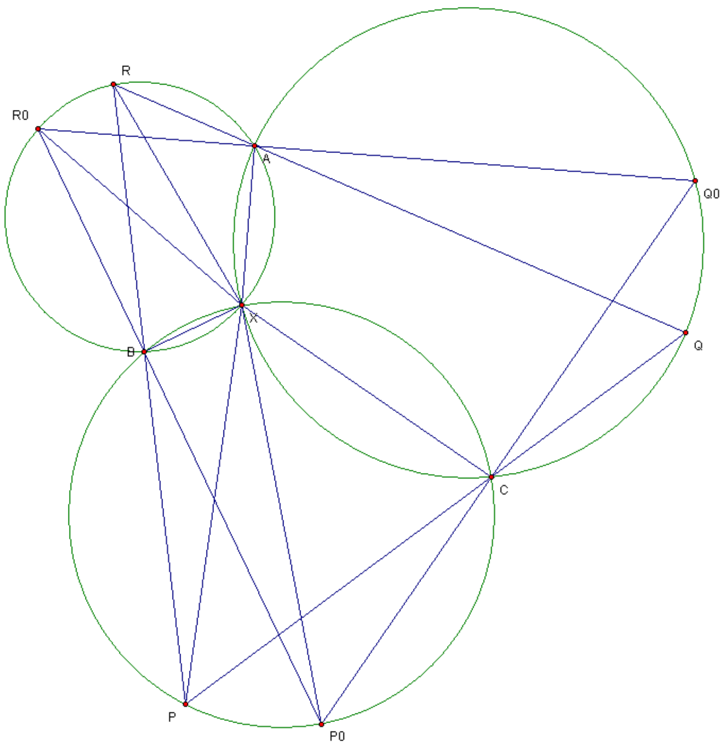
Answer: $\mathbf{2\pi}$

The key is to note that $A_1B_1 + A_3B_3 = A_2B_2 + A_4B_4$ (in terms of arc length). Proof: let P_i be the point of tangency between O and O_i, for $i = 1, 2, 3, 4$. Observe that it is enough to show that

$A_iP_i = B_jP_j$ for all $i, j \in \{1, 2, 3, 4\}$ where $j - i \equiv 1 \pmod 4$. But these two arcs are symmetric with respect to the perpendicular bisector of l_{ij}, as all the small circles have the same radius, so our initial claim is correct.

10. Given a triangle ABC with $BC = 5$, $AC = 7$, and $AB = 8$, find the side length of the largest equilateral triangle PQR such that A, B, C lie on QR, RP, PQ respectively.

Answer: $\mathbf{2\sqrt{43}}$



Let $a = BC$, $b = AC$, $c = AB$. We claim that in general, the answer is $\sqrt{\frac{2}{3}(a^2 + b^2 + c^2 + 4\sqrt{3}S)}$, where S is the area of ABC.

Suppose that PQR is an equilateral triangle satisfying the conditions. Then $\angle BPC = \angle CQA = \angle ARB = 60^\circ$. The locus of points satisfying $\angle BXC = 60^\circ$ is part of a circle O_a. Draw O_b and O_c similarly. These three circles meet at a single point X inside the triangle, which is the unique point satisfying $\angle BXC = \angle CXA = \angle AXB = 120^\circ$. Then the choice of P on O_a determines Q and R: those two points should also be on O_b and O_c respectively, and line segments PCQ and PBR should form sides of the triangle. Now one should find the maximum of PQ under these conditions. Note that $\angle BPX$ and $\angle BRX$ do not depend on the choice of P, so triangle PXR has the same shape regardless of our choice. In particular, the ratio of PX to PR is constant, so PR is maximized when PX is the diameter of O_a. This requires PQ, QR, RP to be perpendicular to XC, XA, XB respectively.

From this point there may be several ways to calculate the answer. One way is to observe that $PQ = \frac{2}{\sqrt{3}}(AX + BX + CX)$ by considering $(PQR) = (PXQ) + (QXR) + (RXP)$. $AX + BX + CX$ can be computed by the usual rotation trick for the Fermat point: rotate $\triangle BXA$ 60° around B to $\triangle BX'A'$. Observe that $\triangle BXX'$ is equilateral, and so A', X', X, and C are collinear. Hence, $A'C = AX + BX + CX$, and we can apply the Law of Cosines to $\triangle A'BC$ to get that $A'C^2 = c^2 + a^2 - 2ac\cos(B + 60^\circ) = a^2 + c^2 + 2ac\sin 60^\circ \sin B - 2ac\cos 60^\circ \cos B = a^2 + c^2 + 2S\sqrt{3} - \frac{1}{2}(a^2 + c^2 - b^2) = \frac{a^2+b^2+c^2}{2} + 2S\sqrt{3} \implies PQ = \sqrt{\frac{2}{3}(a^2 + b^2 + c^2 + 4\sqrt{3}S)}$ (where S is again the area of ABC). Plugging in our values for a, b, and c, and using Heron's formula to find $S = \sqrt{10 * 5 * 3 * 2} = 10\sqrt{3}$, we can calculate $PQ = 2\sqrt{43}$.