# Stanford Math 61DM Pset 0

SUMO

## General Overview

Math 61DM is a course designed to give freshmen a rigorous introduction to linear algebra and discrete math, which includes results about vector spaces, rigorous calculus, combinatorics and graph theory. Although fast-paced, the material is built up from basics, so you don't need to enter in with much except some general knowledge of how to do high-school algebra and a willingness to learn and work with rigorous proof-writing. If you are not (yet!) a confident proof-writer, 61DM offers a 3-week proof-writing session minicourse on the side to help you build and practice that very skill.

## Relevant Notation

All following notation below will be used liberally throughout the course, and is good to be comfortable with beforehand. Most of this is common in virtually all areas of higher math.

- $\forall$ - for all
- $\exists$ - there exists
- $\mathbb{Z}$ - The set of integers $\{\ldots,-1,0,1,\ldots\}$.
- $\mathbb{N}$ - The set of natural numbers $\{0,1,2,\ldots\}$ (if your professor tells you it doesn't contain zero fight them).
- $\mathbb{Q}$ - The set of rational numbers $\{a/b : a,b \in \mathbb{Z}\}$.
- $\mathbb{R}$ - The set of real numbers.
- $x \in X$ - The object $x$ is an element of the set $X$.
- $A \cup B$ - The union of sets $A$ and $B$.
- $A \cap B$ - The intersection of sets $A$ and $B$.
- $A \subseteq B$ - The set $A$ is a subset of the set $B$.
- $A \setminus B$ (also denote $A - B$) - The set of elements of $A$ that are not elements of $B$.
- $\sum_{i=1}^{n} a_i$ - The sum of the objects listed as $a_1, a_2, \ldots a_n$.

Apart from this you should be familiar with standard mathematical notation one would encounter in a high-school algebra or calculus course (e.g functions, addition, less-than signs, etc.).

## Questions

Reminder: It is NOT necessary that you should be know all of these concepts beforehand, or even solve them easily. It is a rare phenomenon in a higher math course to fully solve every single pset.

**Q 1.** Recall for natural numbers $n, k$ with $k \leq n$ that the binomial coefficient $\binom{n}{k}$ (pronounced "$n$ choose $k$") tells us the number of ways to choose $k$ objects out of a set of $n$ objects. It is given by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Prove rigorously the following identities:

$$\frac{n}{k} \times \binom{n-1}{k-1} = \binom{n}{k}$$

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

**Remark.** Why does the identity $1 + 1 = 2$ help solve the final identity above? (If you solved the above using $1 + 1 = 2$, try solving it via induction, and vice versa).

**Q 2.** Prove by induction that:

$$\sum_{i=1}^{n} 3i - 2 = \frac{1}{2} n(3n-1)$$

(If you are shaky on induction or new to it, here's a really good introduction to it, with some elementary examples)

**Q 3.** Define a sequence of real numbers by $x_0 = 1$ and $x_{n+1} = \sqrt{1 + 2x_n}$ for all $n$. Prove that $x_n$ is never greater than 4.

**Definition.** Define a **Group** as a set $G$ with a binary operation $*$ (i.e $*$ is a function that takes two inputs from $G$ and gives an output in $G$) that satisfies:

(i) Identity: $\exists e \in G$ such that $e * g = g * e = g, \ \forall g \in G$.

(ii) Inverses: $\forall g \in G, \ \exists g'$ such that $g' * g = g * g' = e$.

(iii) Associativity: $f * (g * h) = (f * g) * h, \ \forall f, g, h \in G$.

**Q 4.** Using nothing but the core definition of a group given above, show rigorously that inverse elements in a group are unique.

**Hint.** Try using a proof by contradiction for the above problem.

**Q 5.** Either show the following are groups, or explain why they are not:

(i) The set of integers $\mathbb{Z}$, with the binary operation being addition.

(ii) The set of integers $\mathbb{Z}$, with the binary operation being subtraction.

(iii) The set $\mathcal{F}$ of functions $f : \mathbb{R} \to \mathbb{R}$ (i.e with real input and real output), with the operation being multiplication.

(iv) The set of all powers of 2 with the operation being multiplication.

(v) The set $\mathscr{S}$ of all sets[a], with the operation being $\triangle$, which we define as:

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

[a] Technically the "set of all sets" is not actually a set, due to pathological abstract set-theory reasons. More correctly we should be calling the *universe* of all sets, or *collection* of all sets.

**Q 6** (Hard!). Show that the decimal expansion of $(10 + \sqrt{101})^{1000}$ has at least one thousand 9s in it.

**Hint.** Try considering the number $(10 - \sqrt{101})^{1000}$, and use some concept we've already seen in this Pset.