Stanford Math 61CM Pset 0

SUMO

General Overview

Math 61CM is a course designed to give freshmen a rigorous introduction to linear algebra, real anlysis, and multivariable calculus. Although fast-paced, the material is built up from basics, so you don't need to enter in with much except some general knowledge of how to do AP calculus and a willingness to learn and work with rigorous proof-writing. If you are not (yet!) a confident proof-writer, 61CM offers a 3-week proof-writing session minicourse on the side to help you build and practice that very skill.

Relevant Notation

All following notation below will be used liberally throughout the course, and is good to be comfortable with beforehand. Most of this is common in virtually all areas of higher math.

- $\forall$ - for all
- $\exists$ - there exists
- $\mathbb{Z}$ - The set of integers $\{\ldots, -1, 0, 1, \ldots\}$.
- $\mathbb{N}$ - The set of natural numbers $\{0, 1, 2, \ldots\}$ (if your professor tells you it doesn't contain zero fight them).
- $\mathbb{Q}$ - The set of rational numbers $\{a/b : a, b \in \mathbb{Z}\}$.
- $\mathbb{R}$ - The set of real numbers.
- $x \in X$ - The object x is an element of the set X.
- $A \cup B$ - The union of sets A and B.
- $A \cap B$ - The intersection of sets A and B.
- $A \subseteq B$ - The set A is a subset of the set B.
- $A \setminus B$ (also denote $A - B$) - The set of elements of A that are not elements of B.
- $\sum_{i=1}^{n} a_i$ - The sum of the objects listed as $a_1, a_2, \ldots a_n$.

Apart from this you should be familiar with standard mathematical notation one would encounter in a high-school algebra or calculus course (e.g functions, addition, less-than signs, etc.).

Questions

Reminder: It is NOT necessary that you should be know all of these concepts beforehand, or even solve them easily. It is a rare phenomenon in a higher math course to fully solve every single pset.

Q 1. Recall a number is called **rational** if it can be written as a fraction a/b, where a and b are whole numbers, and **irrational** otherwise. Prove by contradiction that $\sqrt{2}$ is irrational.

Remark. Major proof techniques to be aware of are: proof by contradiction, induction, counterexample, and by definition. You do not need to be fully comfortable with working with these, but these will be commonly used throughout 61CM, and there will be many opportunities to work with these proofs!

Q 2. The limit of a function is defined in the following way:

$$\forall\ \epsilon > 0,\ \exists\ \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ whenever } |x - a| < \delta.$$

In this case, we denote:

$$\lim_{x \to a} f(x) = L$$

Using the definition above, prove that:

$$\lim_{x \to 4} \sqrt{x} = 2$$

Q 3. Integrate:

(i) $\int_0^\infty re^{-r^2}\,dr$

(ii) $\int_{\mathbb{R}} \frac{f^{(n)}(x)}{n!}\,dx$ (recall $f^{(n)}$ denotes the n-th derivative of f).

(iii) $\int \frac{1}{x^3+1}\,dx$ (this one is more for fun for those that enjoy integration or want a challenging review).

where f is exponential order (e.g. it can be written in the form $f(x) = g(x)e^{-x}$ for some polynomial g).

Definition. A series $\sum_{n=1}^\infty a_n$ **converges** if the limit of the partial sums $S_N = \sum_{n=1}^N$ is finite, i.e $\lim_{N\to\infty} S_N < \infty$. Otherwise it is said to **diverge**.

Q 4. Prove that the following series converge or diverge (bonus points if you can compute the value if they converge):

(i) $\sum_{n=1}^\infty \frac{1}{n}$

(ii) (Hard!) $\sum_{p \text{ prime}} \frac{1}{p}$

(iii) $\sum_{n=1}^\infty \frac{(-1)^n}{n}$

Q 5. Show that between any two rationals, there exists an irrational, and vice versa.