Stanford Math 210A Pset 0

SUMO

General Overview

Math 210A is the first course in the Stanford graduate department "review series" for the algebra qualifying exams for PhD students. Although aimed at graduate students, much of the material assumes only elementary group, ring and field theory, so it is common for undergraduates to take this course as well. The material covered broadly falls under the umbrella of "commutative algebra", primarily focusing on ring and module theory and homological algebra.

Relevant Notation

All following notation below will be used liberally throughout the course

- $A \times B$ - the direct product of sets, groups, rings, etc.
- A/B - the quotient of some object A (group, ring, etc) by some sub-object B.
- $R^{\times}$ - the set of units (multiplicatively invertible elements) of a ring.
- $R[x]$ - the polynomial ring with coefficents in R and indeterminate x.
- $\phi \circ \psi$ - the composition of functions ψ and ϕ.

Questions

Reminder: It is NOT necessary that you should be know all of these concepts beforehand, or even solve them easily. It is a rare phenomenon in a higher math course to fully solve every single pset.

Q 1. Recall an **ideal** I of a ring R is an additive subgroup of R that is totally closed under multiplication, i.e $ri \in I$ for any $r \in R$ and $i \in I$. One often denotes this $I \trianglelefteq R$.

- Show that any ideal of $R \times S$ is of the form $I \times J$ for $I \trianglelefteq R$ and $J \trianglelefteq S$.
- Show that the equivalence classes of elements of r defined by $r \sim s \iff r - s \in I$ form a ring, called the **quotient ring**, denoted R/I.
- Show that I is not contained in any other proper ideal of R (i.e an ideal that isn't R itself) if and only if the quotient R/I is a field.

Q 2. Suppose a finite group G has even size. Show (from basic group axioms) that G must contain an element of order 2, i.e some $g \in G$ such that $g^2 = e$, where e is the group identity.

Q 3. An ideal I is generated by a set of elements $\{x_n\}$ if I is the smallest ideal containing all the x_n. Show the following two statements are equivalent:

1. Any ideal $I \trianglelefteq R$ is finitely generated. (R is said to be **noetherian** in this case)
2. If $\mathcal{X}$ is any non-empty set of ideals of R then there exists some $I \in \mathcal{X}$ such that no other element of $\mathcal{X}$ contains I.

Hint. For the forward direction, if you have an ascending sequence of ideals $I_1 \subseteq I_2 \subseteq \ldots$, then consider taking their union.

Remark. Most math words named after mathematicians are capitalized by convention, e.g Fourier transform, Eulerian trail, Lagrange point, Gaussian distribution, Dedekind domain, etc. There are two people for whom this is not true: Niels Henrik Abel and Emmy Noether.

Q 4. An ideal is **principal** if it is generated by one element. Which of the following are ideals of $\mathbb{Z}[x]$, and which of those are principal?

- The set of polynomials with constant term that is a multiple of 11.
- The set of polynomial whose x-coefficient is a multiple of 11.
- The set of polynomials whose constant term AND x-coefficient are multiples of 11.

Q 5. If D is a non-zero integer show that:

$$\mathbb{Z}[\sqrt{D}] := \left\{a + b\sqrt{D} : a, b \in \mathbb{Z}\right\}$$

is a subring of $\mathbb{C}$. Show the map $\sigma : \sqrt{D} \mapsto -\sqrt{D}$ induces a ring isomorphism if and only if D is not a perfect square.
In the case that D is not a perfect square, show that $x \in \mathbb{Z}[\sqrt{D}]^\times$ if and only if $x\sigma(x) = \pm 1$.
For $D < 0$ characterize $\mathbb{Z}[\sqrt{D}]^\times$.

Q 6 (Hard!)**.** Suppose $\mathcal{S}$ is a set of rings and $\mathcal{M}$ is a set of homomorphisms (not necessarily every possible one!) between elements of $\mathcal{S}$. A ring R, together with homomorphisms $\phi_S : R \to S$ for every $S \in \mathcal{S}$ is called an **inverse limit** of $(\mathcal{S}, \mathcal{M})$ if:

1. For every $\alpha : S \to T$ in $\mathcal{M}$ we have $\phi_T = \alpha \circ \phi_S$.
2. For any other $(R', \left\{\phi'_S\right\}_{S \in \mathcal{S}})$ satisfying the first condition, there is a UNIQUE map $\psi : R' \to R$ such that $\phi'_S = \phi_S \circ \psi$ for every $S \in \mathcal{S}$.

Show explicitly that every $(\mathcal{S}, \mathcal{M})$ has an inverse limit.

Remark. Sometimes the inverse limit is also just called the limit.