# Stanford Math 171 Pset 0

SUMO

## General Overview

Math 171 is an analysis course that moves beyond Math 115 to cover general metric spaces. For those who have taken Math 115 or 61CM, the class starts bottom-up, with definitions of fields, the reals, sequences and series, then $\varepsilon - \delta$ definitions of limits, continuity, etc. Because of this, content may feel repetitive if you have taken 61CM especially. New topics covered in Math 171 include the basics of measure theory, some convergence theorems, and in general more depth as we spend more time on the topics. Math 171 is also a WiM class.

## Relevant Notation

All following notation below will be used liberally throughout the course, and is good to be comfortable with beforehand. Most of this is common in virtually all areas of higher math.

- $\forall$ - for all
- $\exists$ - there exists
- $\mathbb{Z}$ - The set of integers $\{\ldots, -1, 0, 1, \ldots\}$.
- $\mathbb{N}$ - The set of natural numbers $\{0, 1, 2, \ldots\}$ (if your professor tells you it doesn't contain zero fight them).
- $\mathbb{Q}$ - The set of rational numbers $\{a/b : a, b \in \mathbb{Z}\}$.
- $\mathbb{R}$ - The set of real numbers.
- $x \in X$ - The object $x$ is an element of the set $X$.
- $A \cup B$ - The union of sets $A$ and $B$.
- $A \cap B$ - The intersection of sets $A$ and $B$.
- $A \subseteq B$ - The set $A$ is a subset of the set $B$.
- $A \setminus B$ (also denote $A - B$) - The set of elements of $A$ that are not elements of $B$.
- $\sum_{i=1}^{n} a_i$ - The sum of the objects listed as $a_1, a_2, \ldots a_n$.

Apart from this you should be familiar with standard mathematical notation one would encounter in a high-school algebra or calculus course (e.g functions, addition, less-than signs, etc.).

## Questions

Reminder: It is NOT necessary that you should be know all of these concepts beforehand, or even solve them easily. It is a rare phenomenon in a higher math course to fully solve every single pset.

**Q 1.** Recall the field axioms. Under standard addition and multiplication,

- Prove that the rationals $\mathbb{Q}$ form a field.
- Prove that the integers $\mathbb{Z}$ are not a field.

**Q 2.** Formally state the definition of a limit, i.e. what does it mean if

$$\lim_{x \to a} f(x) = L?$$

Then, prove:

- $\lim_{x \to 4} \sqrt{x} = 2$, and
- $\lim_{x \to \pi/2} \tan(x)$ does not eist.

**Q 3.** State the Archimedean property of the reals, and prove that

$$\lim_{x \to \infty} \frac{1}{x^2} = 0.$$

**Q 4.** State the definition of a supremum, the Least Upper Bound Axiom (or Completeness Axiom) of the reals, and prove that every increasing and bounded sequence of reals $\{a_n\}_{n=1}^{\infty}$ converges.

**Q 5.** Formally state the definition of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at $a \in \mathbb{R}$. Prove that

- $f(x) = \sqrt[3]{x}$ is continuous at 0
- $f(x) = \sin(\frac{1}{x})$ is not continuous at 0
- The pre-image $f^{-1}(A)$ of a continuous $f$ on an open set $A$ is also open.

**Q 6.** Prove that the following series converge or diverge (bonus points if you can compute the value if they converge):

(i) $\sum_{n=1}^{\infty} \frac{1}{n}$

(ii) (Hard!) $\sum_{p \text{ prime}} \frac{1}{p}$

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$