Math 120 Pset 0

SUMO

September 2024

General Overview

Math 120 is an introductory course on objects called groups and some topics related to objects called rings. The bulk of the course focuses on groups, while the last two to three weeks focuses on rings. The biggest thing you should note is that, even though this is intended to be an introductory proof-writing course, a lot of the problem sets are notoriously bad at hand-holding through proofs, so it is a good idea to be familiar with the structure of mathematical proofs beforehand (induction, contradiction, contrapositive and the language of math like "implies", "iff"). This is also a WIM course. The instructor usually gives around three options for the WIM paper, which is written towrad the middle to late portion of the course. Previous topics have included semidirect products, the projective special linear group, and finite simple groups among others. This is usually not too intensive, although you will still have a problem set during the week the WIM is due.

Relevant Notation

All following notation below will be used liberally throughout the course

- $A \times B$ - the direct product of groups, sets, rings, objects etc.
- A/B - the quotient of A by B, whether they be groups or rings.
- φ - Euler's totient function, the number of integers at most n coprime to n.
- $f \circ g$ - the composition of f and g, where g is performed first
- $f : A \to B$ - f defines a map (homomorphism) of elements of A to elements of B.
- $a \in A$ - a is an element of A.
- $\cdot, +$ - group operations written multiplicatively and additively, resp., depending on context.
- $\mathbb{Z}$ - the integers $\ldots, -2, -1, 0, 1, 2, \ldots$.
- $\mathbb{R}$ - the real numbers.

Questions

Reminder: It is NOT necessary that you should be know all of these concepts beforehand, or even solve them easily. It is a rare phenomenon in a higher math course to fully solve every single pset.

Q 1. Let $G = \{1,2,3,4\}$ be a set, and define a function $\cdot : G \times G \to G$ which takes elements $a, b \in G$ and produces an element $a \cdot b \in G$. For example, we could have the following times table which defines one such $\cdot$

$\cdot$	1	2	3	4
1	1	2	3	4
2	2	1	4	2
3	3	1	4	3
4	4	4	3	2

In this example, $2 \cdot 3 = 4$ and $3 \cdot 2 = 1$.

- Find a function $\cdot$ such that $a \cdot b \neq b \cdot a$ for all $a, b \in G$ (hint: look at the given example)
- Find a function $\cdot$ such that $(a \cdot b) \cdot c \neq a \cdot (b \cdot c)$ for all $a, b, c \in G$.
- Find a function $\cdot$ such that there is no element $e \in G$ for which $e \cdot b = b \cdot e = b$ for all $b \in G$.
- Find a function $\cdot$ such that $1 \cdot b = b \cdot 1 = b$ for all $b \in G$, but there is some $a \in S$ for which there is no "inverse", i.e. there is no $b \in G$ with $a \cdot b = b \cdot a = 1$.

Remark. These outline the axioms of a group. A group is a set G together with a binary operation $\cdot : G \times G \to G$ (i.e. multiplication or addition depending on your point of view) such that three things hold:

1. The operation is associative, i.e. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.
2. There is an "identity" $e \in G$, an element like "1" for multiplication or "0" for addition such that $e \cdot a = a \cdot e = a$ for all $a \in G$.
3. For every $a \in G$, there is an "inverse" $b \in G$ like "1/a" for multiplication or "-a" for addition such that $a \cdot b = b \cdot a = e$. The inverse is usually written a^{-1}, i.e. $a \cdot a^{-1} = a^{-1} \cdot a = e$.

We say that "*G is a group under $\cdot$*" if it is a set with binary operation $\cdot$ satisfying the above axioms.

Q 2. Let G be a group (see the above remark for a definition).

- Show that $\mathbb{Z}$ is a group under addition. That is, we have an operation $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by regular addition of integers $a + b$ (i.e. $2 + 3 = 5$, what you expect). Show that it satisfies the three axioms above (you may take associative for granted).
- Show that $\mathbb{R} \setminus 0$, the real numbers except for 0, form a group under multiplication (again, take associative for granted).
- Show that e is *unique*. That is, if e' is another element such that $e' \cdot a = a \cdot e' = a$ for all $a \in G$, then $e' = e$.
- Show that a^{-1} is unique. If b is another element such that $a \cdot b = b \cdot a = e$, then $b = a^{-1}$.

Remark. Recall that a *bijection* is a function of sets $f : X \to X$ that is one-to-one (injective) and onto (surjective). That is, if $f(a) = f(a')$ then $a = a'$ (injective) and for every $b \in X$ there is some $a \in X$ such that $f(a) = b$. For example, if $X = \{1,2,3,4\}$ and $f(1) = 2, f(2) = 3, f(3) = 4, f(4) = 1$ then f is a bijection since every element gets mapped to by exactly one element. However, if $f(1) = 2, f(2) = 2, f(3) = 4, f(4) = 1$ then f is not injective since $f(1) = f(2)$ but $1 \neq 2$, and is not surjective since nothing maps to 3.

Q 3. Fix a finite set X (for example $X = \{1,2,3,4\}$ as above). Show that the set S_X of bijections $f: X \to X$ is a group under function composition.

Q 4. Let $X = \{1,2,\ldots,n\}$ where $n \geq 1$ is some integer. In this case, we let S_n denote the group of bijections $f: X \to X$. Then S_n is called *the symmetric group on n elements*.

- How many elements are in S_2? S_3?
- How many elements are in S_n?
- Find elements $f, g \in S_n$ such that $f \circ g \neq g \circ f$.

Remark. S_n is one of the most important examples of a group since it appears everywhere, especially Galois theory and representation theory.

Q 5. This is more open-ended, but give axioms for what you intuitively think a subgroup H of a group G should be.

Hint. A subgroup H of a group G is a subset of G that is also a group under the given operation. Can you think of a concise criterion for H to be a subgroup?