1. Answer: \(\frac{-1}{x^2+1}\)

Notice that as \(t \to 0\), both the numerator and the denominator approach 0. Thus, applying L’Hôpital’s rule on \(t\) (keeping \(x\) constant):

\[
\frac{d}{dt} \left[ \tan^{-1} \left( \frac{1}{x + t} \right) \right]_{t=0} = -\frac{1}{1 + x^2}
\]

2. Answer: 1

Let \(f(x) = e^x - x - \frac{x^2}{2}\). Then \(f'(x) = e^x - 1 - x^2\). When \(x < 0\), \(e^x < 1\) and \(1 + x^2 > 1\), so \(f'(x) = e^x - (1 + x^2) < 0\). Thus, \(f\) is decreasing on \((-\infty, 0)\). When \(x = 0\), \(f'(x) = f'(0) = e^0 - 1 - 0^2 = 1 - 1 = 0\). Finally, for \(x > 0\), \(f'(x) = e^x - 1 - x^2 > 0\) by a Maclaurin series expansion, so \(f\) is increasing on \((0, \infty)\).

Thus, \(f\) must attain its minimum when \(x = 0\), at which point \(f\) has the value \(e^0 - 0 - \frac{0^2}{2} = 1\).

3. Answer: \(\sqrt{2}\)

Consider:

\[
\frac{d}{dt} \sin^{-1}(t - \sqrt{1/2}) \bigg|_{t=0} = \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f(x) \, dx \bigg|_{t=0} = \int_{-\infty}^{\infty} xe^{tx} f(x) \, dx \bigg|_{t=0} = \int_{-\infty}^{\infty} xf(x) \, dx
\]

\[
\frac{d}{dt} \sin^{-1}(t - \sqrt{1/2}) \bigg|_{t=0} = \frac{1}{\sqrt{1 - \left(\sqrt{1/2} - t\right)^2}} \bigg|_{t=0} = \frac{1}{\sqrt{1 - (1/2)}} = \sqrt{2}.
\]

4. Answer: \(x = -\frac{2}{3}\) and \(x = 0\)

Notice that \(f(x) \to 0\) as \(x \to \pm \infty\). Since \(9x^2 + 6x + 2\) has no real roots, the maximum value of \(f(x)\) is attained at the maximum of the absolute values of the critical points of \(\frac{3x+1}{9x^2+6x+2}\).

The extrema of \(\frac{3x+1}{9x^2+6x+2}\) occur at \(x = -\frac{2}{3}\) and \(x = 0\). It is easily checked that maxima of \(f(x)\) occur at both of these points.

5. Answer: \(\frac{128\sqrt{3}}{27}\)

Let the circular island be a circle of radius 2 centered at the origin. Without loss of generality, let the length of the rectangular base be from \(-x\) to \(x\) and the width from \(-y\) to \(y\). Notice that by the equation of a circle, \(x^2 = 4 - y^2\). Then

\[
V = \frac{1}{3} (2x)^2 (2y) = \frac{8}{3} x^2 y = \frac{8}{3} (4 - y^2) y = \frac{8}{3} (4y - y^3)
\]

\[
\frac{dV}{dy} = \frac{8}{3} (4 - 3y^2) = 0 \implies y = \frac{\sqrt{4}}{\sqrt{3}} = \frac{2}{\sqrt{3}}.
\]

\[
V = \frac{8}{3} \left(\frac{2}{\sqrt{3}}\right) \sqrt{\frac{4}{3}} = \frac{128\sqrt{3}}{9\sqrt{3} \cdot 27} = \frac{128\sqrt{3}}{27}.
\]

6. Answer: 13

This is the evaluation of the mean of a Poisson distribution: for any \(\lambda\),

\[
\sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} = \lambda e^{-\lambda} e^\lambda = \lambda.
\]

7. Answer: \(-\frac{2\cos(t^2)}{t}\)
By the Leibniz integral rule, the above integral becomes

\[ \int_{-\ln 1/t}^{\ln 1/t} -e^x \sin(te^x) \, dx + \cos(te^{\ln(1/t)})(-1/t) - \cos(te^{-\ln(1/t)})(1/t) \]

\[ = \frac{\cos(te^x)}{t} \bigg|_{-\ln 1/t}^{\ln 1/t} - \frac{\cos(1) + \cos(t^2)}{t} \]

\[ = -2 \cos(t^2) \]

8. **Answer: \( \ln 3 \)**

The partial sums of this sum are equal to

\[ \left( \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{3n} \right) - 3 \left( \frac{1}{3 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{3 \cdot n} \right) \]

\[ = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n} = \frac{1}{n} \left( \frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{3n}{n}} \right) \]

This is a Riemann sum, so as \( n \to \infty \) the partial sums converge to

\[ \int_1^2 \frac{1}{1+x} \, dx = \ln 3. \]

9. **Answer: 3**

As you can see it from this graph, \( F(k) \) is the area of region that “lies between” \( y = f(x) = x(4-x) \) and \( y = k \). Let \( A \) be the region below \( y = f(x) \) and above \( y = k \), and \( B \) be the region below \( y = k \) and above \( y = f(x) \). Then \( F(k) = A + B \). Meanwhile, we can find the area of \( A \) by integrating with respect to \( y \)-variable. Since \( x \) belongs to the interval of length \( l(t) \) when \( y = t \), we can say

\[ A = \int_k^4 l(t) \, dt. \]

Apply the same reasoning to \( B \), then we have

\[ B = \int_0^k (4 - l(t)) \, dt. \]

Thus, by the fundamental theorem of calculus,

\[ \frac{d}{dk} F(k) = \frac{d}{dk} A + \frac{d}{dk} B \]

\[ = \frac{d}{dk} \left( \int_k^4 l(t) \, dt \right) + \frac{d}{dk} \left( \int_0^k (4 - l(t)) \, dt \right) \]

\[ = -l(k) + (4 - l(k)) \]

\[ = 4 - 2l(k). \]
Since \( l(k) \) is decreasing by \( k \), \( F(k) \) achieves minimum when \( \frac{d}{dk} F(k) = 0 \). One can easily find that \( k = 3 \) if \( l(k) = 2 \), so the answer is 3.

10. **Answer:** \( y = -4x^2 + 5x - 7 \)

Such a parabola intersects \( f(x) \) precisely where \( f'(x) = 0 \). Hence, the value of the intersection points do not change when we replace \( f(x) \) by \( f(x) + g(x)f'(x) \) for any \( g(x) \). Therefore, since \( f'(x) = 6x^5 - 12x + 6 \), we must have that \( f(x) - 1/6xf'(x) = -4x^2 + 5x - 7 \) passes through the three critical points. Since three points determines a parabola uniquely, this must be the unique parabola passing through the three critical points.