1. Answer: $\frac{\sqrt{5} - 1}{4}$.

Consider:

$$\sin 18^\circ = \cos 72^\circ = 2 \cos^2 36^\circ - 1 = 2(1 - 2 \sin^2 18^\circ)^2 - 1$$

$$= 1 - 8 \sin^2 18^\circ + 8 \sin 18^\circ$$

$$0 = 8x^4 - 8x^2 - x + 1 = (x - 1)(2x^2 + 2x + 1).$$

Clearly, $x \neq 1, \frac{-1}{2}$, because $0 < \sin 18^\circ < \sin 90^\circ = 1$. We solve the remaining term:

$$0 = 4x^2 + 2x + 1 \implies x = \frac{-2 \pm \sqrt{4 + 4(4)}}{2(4)} = \frac{-1 \pm \sqrt{5}}{4}.$$

The only root that is within our bounds is $\frac{\sqrt{5} - 1}{4}$.

2. Answer: $\frac{44}{15} + \frac{4}{15}i$.

Consider:

$$30123 = 3 + \sum_{n=0}^{\infty} 0(2i)^{-(4n+1)} + \sum_{n=0}^{\infty} 1(2i)^{-(4n+2)} + \sum_{n=0}^{\infty} 2(2i)^{-(4n+3)} + \sum_{n=0}^{\infty} 3(2i)^{-(4n+4)}$$

$$= 3 + \sum_{n=0}^{\infty} \left[ -\frac{1}{4} \left( \frac{1}{16} \right)^n + \frac{i}{4} \left( \frac{1}{16} \right)^n + \frac{3}{16} \left( \frac{1}{16} \right)^n \right]$$

$$= 3 + \sum_{n=0}^{\infty} \left( -\frac{1}{16} + \frac{1}{4}i \right) \left( \frac{1}{16} \right)^n$$

$$= 3 + \frac{16}{15} \left( -\frac{1}{16} + \frac{1}{4}i \right)$$

$$= \frac{44}{15} + \frac{4}{15}i.$$


This is a trivial application of Ramsey Theory. It is isomorphic to Problem 2 of the Power Round. Consider one of the people, $P$, in the group, and that he or she may or may not have met last year. Assume without loss of generality that $P$ met at least three of them last year: $A$, $B$, and $C$. If any two of these met each other last year, then those two and $P$ all met each other last year. Alternatively, none of $A$, $B$, and $C$ met each other last year.

4. Answer: $\frac{1}{16} \cos 5\theta + \frac{5}{16} \cos 3\theta + \frac{5}{8} \cos \theta$

Notice that

$$\cos(n\theta) + i \sin(n\theta) = (\cos \theta + i \sin \theta)^n = z^n$$

and

$$\cos(-n\theta) + i \sin(-n\theta) = (\cos \theta + i \sin \theta)^{-n} = z^{-n}.$$.

Adding these two equations, we get that $\cos(n\theta) = (z^n + z^{-n})/2$. Then $(\cos(\theta))^5 = (z + z^{-1})^5/32$.

Expanding yields the binomial coefficients:

$$(z + z^{-1})^5 = z^5 + 5z^4(z^{-1}) + 10z^3(z^{-2}) + 10z^2(z^{-3}) + 5z(z^{-4}) + z^{-5}.$$.

Then

$$(z + z^{-1})^5/32 = \frac{1}{16}(z^5 + z^{-5})/2 + \frac{5}{16}(z^3 + z^{-3})/2 + \frac{5}{8}(z + z^{-1})/2 = \frac{1}{16} \cos 5\theta + \frac{5}{16} \cos 3\theta + \frac{5}{8} \cos \theta.$$
5. **Answer:** $2^{2011} - 1$

   We evaluate the inner sum by the Hockey Stick Identity. This identity is
   \[
   \sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1} \implies \sum_{i=j}^{2010} \binom{i}{j} = \binom{2011}{j+1},
   \]
   so that
   \[
   \sum_{j=0}^{2010} \sum_{i=j}^{2010} \binom{i}{j} = \sum_{j=0}^{2010} \binom{2011}{j+1}.
   \]
   Now, using the fact that
   \[
   \sum_{i=0}^{n} \binom{n}{i} = 2^n,
   \]
   we obtain
   \[
   \sum_{j=0}^{2010} \binom{2011}{j+1} = \sum_{j=1}^{2011} \binom{2011}{j} = \sum_{j=0}^{2011} \binom{2011}{j} - \binom{2011}{0} = 2^{2011} - 1.
   \]

6. **Answer:** 96

   The number of blue cells is $n + m - 1$; the number of total cells is $nm$. So $2010(m + n - 1) = nm$, or $nm - 2010n - 2010m + 2010 = 0$. This factors as $(n - 2010)(m - 2010) - 2010^2 + 2010 = 0$, or $(n-2010)(m-2010) = 2010-2009$. Thus each of $n-2010$ and $m-2010$ must be one of the positive factors of $2010 + 2009$; for each positive factor, there is one ordered pair. Since $2010 \cdot 2009 = 2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 41 \cdot 67$, there are $2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$ solutions.

7. **Answer:** $\frac{1}{p} - 1$

   Let the probability that a bug’s descendant’s die out be $x$. There are two ways for the bugs to die out: either the initial bug dies (with probability $1-p$), or the bug successfully splits (probability $p$) and both of its descendants die out (each with probability $x$). Therefore, $x = (1-p) + px^2$. Solving this quadratic equation yields the two solutions $x = 1$ and $x = \frac{1}{p} - 1$. Which is correct?

   Define $p_n$ to be the probability that the bug dies out within $n$ generations. Then, by the same reasoning as before, $p_{n+1} = (1-p) + pp_n^2$. From the definition of $p_n$, we see that the sequence is always increasing. We will show that $p_n < \frac{1}{p} - 1$ for every $n$, which would imply that $x = \frac{1}{p} - 1$ is the correct solution.

   This can be done by induction. Notice that $p_0 = 0 < \frac{1}{p} - 1$. Now, suppose that $p_k < \frac{1}{p} - 1$ for some $k$. Then,
   \[
   p_{k+1} = (1-p) + pp_k^2 < (1-p) + p\left(\frac{1}{p} - 1\right)^2 = 1-p + p\left(\frac{1}{p^2} - 2\frac{1}{p} + 1 \right) = 1-p + \frac{1}{p} - 2 + p = \frac{1}{p} - 1.
   \]
   This completes the induction, so we indeed have $p_n < \frac{1}{p} - 1$, and hence the correct answer is indeed $\frac{1}{p} - 1$.

8. **Answer:** $(1, 1)$ and $(3, 2)$

   From that $3^1 \equiv 3$, $3^2 \equiv 9$, $3^3 \equiv 11$, $3^4 \equiv 1 \mod 16$, we can see that $3^y \not\equiv 1 \mod 16$. Thus $2^x \not\equiv 0 \mod 16$, so $x$ is at most 3. The solutions are $(1, 1)$ and $(3, 2)$.

9. **Answer:** 4

   If $(x, y)$ is a solution, then $x$ and $y$ are roots of the quadratic equation $t^2 + at + b = (t-x)(t-y) = 0$. Then $-a = x + y$ and $b = xy$. We find that $x^2 + y^2 = a^2 - 2b = 9$ and $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{-b}{9} = 9$. So $a = -9b$. Substituting this in the first equation, we get $81b^2 - 2b - 9 = 0$. This has two roots for $b$, both of them real. Therefore there are two corresponding values of $a$, both real. In each case, the quadratic leads to two ordered pairs, which gives four total ordered pairs. It is easy to check that they are, indeed, distinct.
10. **Answer:** 120

Note that \( n^2 \equiv 0, 1, 4 \mod 5 \). We consider three cases.

**Case 1:** \( n^2 \equiv 0 \mod 5 \), so that \( \lfloor \frac{n^2}{5} \rfloor = \frac{n^2}{5} \). In this case, \( n \equiv 0 \mod 5 \), so \( n = 5a \) for some integer \( a \). Then \( \frac{n^2}{5} = 5a^2 \), which is not prime unless \( a = 1 \). Therefore, for this case, \( n = 5 \) is the only value of \( n \) for which \( \lfloor \frac{n^2}{5} \rfloor \) is prime.

**Case 2:** \( n^2 \equiv 1 \mod 5 \), so that \( \lfloor \frac{n^2}{5} \rfloor = \frac{n^2 - 1}{5} = \frac{(n - 1)(n + 1)}{5} \). In this case, we have either \( n = 5a + 1 \) or \( n = 5a - 1 \) for some integer \( a \). Then \( \frac{n^2}{5} = a(n \pm 1) \), which cannot be prime if \( a \neq 1 \). Therefore, for this case, \( n = 4, 6 \) are the only values of \( n \) for which \( \lfloor \frac{n^2}{5} \rfloor \) might be prime. We can check that these values of \( n \) do indeed yield primes 3 and 7.

**Case 3:** \( n^2 \equiv 4 \mod 5 \), so that \( \lfloor \frac{n^2}{5} \rfloor = \frac{n^2 - 4}{5} = \frac{(n - 2)(n + 2)}{5} \). In this case, we have either \( n = 5a + 2 \) or \( n = 5a - 2 \) for some integer \( a \). Then \( \frac{n^2}{5} = a(n \pm 2) \), which cannot be prime if \( a \neq 1 \). Therefore, for this case, \( n = 3, 7 \) are the only values of \( n \) for which \( \lfloor \frac{n^2}{5} \rfloor \) might be prime. None of these values actually yield primes however, as they give \( \lfloor \frac{n^2}{5} \rfloor = 1, 9 \).

Therefore, the only values of \( n \) for which \( \lfloor \frac{n^2}{5} \rfloor \) is prime are \( n = 4, 5, 6 \), and the product of these values of \( n \) is 120.

**Generalization:** Note that this procedure can be carried out when the denominator 5 is replaced by any other number whose quadratic residues are all perfect squares. Which numbers satisfy this property?