1. Let \(a, b, c,\) and \(d\) be the numbers that show when four fair dice, numbered 1 through 6 are rolled. What is the probability that \(|(a-1)(b-2)(c-3)(d-6)| = 1\)?

**Answer:** \(\frac{1}{324}\)

The conditions imply that \(|a-1| = |b-2| = |c-3| = |d-6| = 1\). \(a\) can equal 2, \(b\) can equal 1 or 3, \(c\) can equal 2 or 4, and \(d\) can equal 5. So the probability is \(\frac{1}{6} \times \frac{2}{6} \times \frac{2}{6} \times \frac{1}{6} = \frac{1}{324}\).

2. Find all possibilities for the second-to-last digit of a number whose square is of the form 1,2,3,4,5,6,7,8,9,0 (each _ is a digit).

**Answer:** 3, 7

Zero is the only digit with square ending in 0. The square of a number ending in zero will therefore end in two zeros. Next digit of the number therefore needs a square ending in 9, so it is 3 or 7.

3. Ten gears are lined up in a single file and meshed against each other such that the \(i^{th}\) gear from the left has \(5i+2\) teeth. Gear 1 (counting from the left) is rotated 21 times. How many revolutions does gear 10 make?

**Answer:** \(\frac{147}{52}\)

The number of teeth meshed does not vary. Thus, if \(n\) is the number of revolutions that gear 10 make, then \((5(1) + 21)(21) = (5(10) + 2)n \Rightarrow n = \frac{7\times 21}{52} = \frac{147}{52}\).

4. In the game Pokeymawn, players pick a team of 6 different Pokeymawn creatures. There are 25 distinct Pokeymawn creatures, and each one belongs to exactly one of four categories: 7 Pokeymawn are plant-type, 6 Pokeymawn are bug-type, 4 Pokeymawn are rock-type, and 8 Pokeymawn are bovine-type. However, some Pokeymawn do not get along with each other when placed on the same team: bug-type and bovine-type Pokeymawn will eat anything except other Bovines. How many ways are there to form a team of 6 different Pokeymawn such that none of the Pokeymawn on the team want to eat any of the other Pokeymawn?

**Answer:** 245

If we make our team all the same type, then there are \(\binom{7}{6} + \binom{6}{6} + \binom{4}{6} = 7 + 1 + 0 + 28 = 36\) ways to do this. If we make our team partially bug and partially rock type, there are \(\binom{6}{4}) + \binom{6}{4} \binom{4}{1}) + \binom{6}{4} \binom{2}{1} + \binom{6}{4} \binom{3}{1} = 15 \times 1 + 20 \times 4 + 15 \times 6 + 6 \times 4 = 15 + 80 + 90 + 24 = 209\) ways. Any other combination of types will not work. This gives a total of 245 ways.

5. Four cards are drawn from a standard deck (52 cards) with suits indistinguishable (for example, A♠ is the same as A♥). How many distinct hands can one obtain?

**Answer:** 1820, or \(\binom{13}{1} + 3\binom{13}{2} + 3\binom{13}{3} + \binom{13}{4}\)

We proceed by casework.

**Case 1** All cards have the same face value. There are \(\binom{13}{1}\) ways to choose the face values.

**Case 2** Some cards have face value \(A\); some have face value \(B\). There are \(\binom{13}{2}\) ways to choose \(A\) and \(B\). One can have the combinations \(ABBB, AABB, AAAB,\) so there are \(3\binom{13}{2}\) distinct ways for this case.

**Case 3** Some cards have face value \(A\), some \(B\), and some \(C\). There are \(\binom{13}{3}\) ways to choose the \(A, B, C\). One can have the combinations \(ABCC, ABBC,\) and \(AABC\). There are \(3\binom{13}{3}\) distinct ways for this case.

**Case 4** The cards are distinct: \(ABCD\). There are \(\binom{13}{4}\) ways to do this. Since these cases are mutually exclusive, we have \(\binom{13}{1} + 3\binom{13}{2} + 3\binom{13}{3} + \binom{13}{4}\) = 1820 distinct hands.

6. Find all complex numbers \(z\) such that \(z^5 = 16\bar{z}\), where if \(z = a + bi\), then \(\bar{z} = a - bi\).

**Answer:** \(0, \pm 2, 1 \pm i\sqrt{3}, -1 \pm i\sqrt{3}\)
10. Evaluate \( e^{\pi/3} + 2e^{2\pi/3} + 2e^{3\pi/3} + 2e^{4\pi/3} + e^{5\pi/3} + 9e^{6\pi/3} \).

**Answer:** 6

\( e^{\pi/3} + e^{2\pi/3} + e^{3\pi/3} + e^{4\pi/3} + e^{5\pi/3} + e^{6\pi/3} \) sum to 0 because the terms are sixth roots of unity (i.e. they satisfy \( z^6 - 1 = 0 \), which is a 6th degree polynomial whose 5th degree coefficient is 0). Likewise, \( e^{2\pi/3} + e^{4\pi/3} + e^{6\pi/3} \) sum to zero because the terms are cubic roots of unity. \( e^{3\pi/3} + e^{6\pi/3} \) sum to 0 because they are square roots of unity. Subtracting these sums from the original expression, we are left with only 6\( e^{6\pi/3} \), which is 6(cos(2\pi) + i sin(2\pi)) = 6.