

CALCULUS TEST
2007 STANFORD MATH TOURNAMENT
MARCH 4, 2007

1. Find

$$\lim_{x \rightarrow 0} \frac{-1 + \cos x}{3x^2 + 4x^3}$$

2. A line through the origin is tangent to $y = x^3 + 3x + 1$ at the point (a, b) . What is a ?
3. A boat springs a leak at time $t = 0$, with water coming in at constant rate. At a time $t = \tau > 0$ hours, someone notices that there is a leak and starts to record distance the boat travels. The boat's speed is inversely related to the amount of water in the boat. If the boat travels twice as far in the first hour as in the second hour, what is τ ?
4. Let $I(n) = \int_0^\pi \sin(nx) dx$. Find $\sum_{n=0}^{\infty} I(5^n)$.
5. Let $\Theta_k(x)$ be 0 for $x < k$ and 1 for $x \geq k$. The Dirac delta "function" is defined to be $\delta_k(x) = \frac{d}{dx} \Theta_k(x)$. (It's really called a distribution, and we promise it makes sense.) Suppose $\frac{d^2}{dx^2} f(x) = \delta_1(x) + \delta_2(x)$ and $f(0) = f'(0) = 0$. What is $f(5)$?
6. Point A is chosen randomly from the circumference of the unit circle, while point B is chosen randomly in the interior. A rectangle is then constructed using A, B as opposite vertices, with sides parallel or perpendicular to the coordinate axes. What is the probability that the rectangle lies entirely inside the circle?
7. A balloon in cross-section has the equation $y = \pm\sqrt{2x - x^2}e^{-x/2}$, with the x -axis beginning at the top of the balloon pointing toward the knot at the bottom. What is its volume?
8. Silas does nothing but sleep, drink coffee, and prove theorems, and he never more than one at a time. It takes 5 minutes to drink a cup of coffee. When doing math, Silas proves $s + \ln c$ theorems per hour, where c is the number of cups of coffee he drinks per day, and s is the number of hours he sleeps per day. How much coffee should Silas get in a day to prove the most theorems?
9. Evaluate $\lim_{n \rightarrow \infty} \sum_{k=n+1}^{2n} \frac{1}{k}$.
10. Find the 10^{th} nonzero term of the power series for $f(x) = \frac{x}{(x^2-1)^2}$ (expanding about $x = 0$).