1. Answer: $-\frac{1}{6}$

Use l’Hopital’s rule:

$$\lim_{x \to 0} \frac{-1 + \cos x}{3x^2 + 4x^3} = \lim_{x \to 0} \frac{-\sin x}{6x + 12x^2} = \lim_{x \to 0} \frac{-\cos x}{6 + 12x}$$

2. Answer: $\frac{3\sqrt{3}}{2}$

$$y' = 3a^2 + 3 = \frac{a^3 + 3a + 1}{a} = \frac{y}{x}$$

$$\frac{2a^3 - 1}{a} = 0$$

$$a^3 = \frac{1}{2}$$

3. Answer: $\frac{\sqrt{5} - 1}{2}$

The speed will cancel out so assume it is 1. We then have:

$$\int_\tau^{\tau + 1} \frac{1}{t} \, dt = 2 \int_{\tau + 1}^{\tau + 2} \frac{1}{t} \, dt$$

$$\ln \frac{\tau + 1}{\tau} = 2 \ln \frac{\tau + 2}{\tau + 1}$$

$$\frac{\tau + 1}{\tau} = \left( \frac{\tau + 2}{\tau + 1} \right)^2$$

$$\tau = \frac{-1 \pm \sqrt{5}}{2}$$

4. Answer: $\frac{5}{2}$

For odd $n$, $I(n) = -\frac{\cos(nx)}{n} \bigg|_0^\pi = 2/n$, so $\sum_{n=0}^\infty I(5^n) = \sum_{n=0}^\infty 2/5^n = 5/2$

5. Answer: 7

We have $f'(x) = \int (\delta_1(x) + \delta_2(x)) \, dx = \Theta_1(x) + \Theta_2(x) + C$, and $f'(0) = 0$ so $C = 0$. Integrating up to $f$ is most easily accomplished graphically; the region under the curve from 0 to 5 is a $1 \times 4$ rectangle from $x = 1$ to $x = 5$ with a $1 \times 3$ rectangle from $x = 2$ to $x = 5$ on top.

6. Answer: $\frac{4}{\pi^2}$

Suppose $A$ lies at polar coordinate $0 < \theta < \pi/2$. For the rectangle to lie within the circle, $B$ must lie in the rectangle with vertices at $A$, $A$ reflected over the $x$-axis, $A$ reflected over the $y$-axis, and $A$ reflected over both axes. Thus for this fixed $A$, the probability is $(2\sin \theta)(2\cos \theta)/\pi = 2\sin(2\theta)/\pi$. The total probability is then $\frac{2}{\pi} \int_0^{\pi/2} 2 \sin(2\theta) \, d\theta$. (Integrating over the circle requires taking the absolute value of the expression for area, which then splits up into four sections identical to the one considered here.)
7. Answer: \( \frac{4\pi}{e^2} \)

\[
V = \pi \int_0^2 \left( \sqrt{2x - x^2} e^{-x/2} \right)^2 dx = \pi \int_0^2 (2x - x^2)e^{-x} dx = \pi x^2 e^{-x} \bigg|_0^2
\]

8. Answer: 12 cups of coffee

The number of theorems proven is \((s + \ln c)(24 - s - c/12)\). Differentiating with respect to \(s\) gives \(24 - \frac{c}{12} - 2s - \ln c = 0\), so \(s = 12 - \frac{c}{24} - \frac{1}{2} \ln c\). This is a maximum in \(s\) since the second derivative is \(-2\). Plugging this back in and simplifying gives \((12 - \frac{c}{24} + \frac{\ln c}{2})^2 = f(c)^2\) theorems proven. This differentiates to \(2f'(c)f(c)\), so the derivative will be zero when either \(f(c)\) or \(f'(c)\) is zero. \(f(c) = 0\) is difficult to solve, involving both a logarithm and a binomial, but \(f'(c) = \frac{1}{24} - \frac{1}{24}\), so \(c = 12\) is a solution. It is a maximum in \(c\) since the second derivative is \(2f''(c)^2 + 2f(c)f''(c)\), with \(f''(12) < 0\), \(f(12) > 0\), and \(f'(12) = 0\).

9. Answer: \(\ln 2\)

\[
\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{n}{n} \sum_{k=1}^{n} \frac{1}{k+n} = \sum_{k=1}^{n} \frac{1}{n} \frac{1}{1 + \frac{k}{n}}
\]

This is a Riemann sum: \(\int_1^{\frac{1}{2}} x^n \,dx = \ln 2\).

10. Answer: \(10x^{19}\)

Note that \(\int f(x)\,dx = \frac{1}{2(1-x)} = \frac{1}{2} \left( \frac{1}{1+x} + \frac{1}{1-x} \right)\). These are geometric sums, so we have

\[
\int f(x)\,dx = \frac{1}{4} \left( \sum_{k=0}^\infty x^k + \sum_{k=0}^\infty (-x)^k \right)
\]

\[
= \frac{1}{2} \sum_{k=0}^\infty x^{2k}
\]

\[
f(x) = \sum_{k=0}^\infty kx^{2k-1}
\]