1. Answer: 29

\[140\pi = \text{volume of cone } M - \text{volume of cone } N = \frac{1}{3} \cdot x^2 \cdot 20\pi - \frac{1}{3} \cdot x \cdot (20)^2 \pi = \frac{20x^2\pi}{3} - \frac{400x\pi}{3}\]
\[20x^2 - 400x = 420 \Rightarrow x^2 - 20x - 21 = 0 \Rightarrow (x - 21)(x + 1) = 0 \Rightarrow x = 21, -1\]
But \(x\) must be positive, so \(x = 21\).

\[BC = \sqrt{AB^2 + AC^2} = \sqrt{20^2 + 21^2} = 29\]

2. Answer: \(-2\)

Let the first element be \(x\), and the second, \(y\). Writing out each element in terms of \(x\) and \(y\) gives \(\{x, y, 2x + y, 5x + 3y, 13x + 8y, \ldots\}\), which is apparently the fibonacci sequence with every other element as the coefficient of \(x\) or \(y\). So the 6th element is \(34x + 21y\) and the seventh, \(89x + 55y\). Solving \(89 \cdot 2 + 55 \cdot y = 68\) gives \(y = -2\).

3. Answer: 17.5

Form \(\triangle ABC\), and set \(a = BC, b = AC,\) and \(c = AB\). Let 5 be the altitude from \(A, 7\) be the altitude from \(B,\) and call the third altitude \(h\).

\[5a = 7b = h \cdot c, \text{ so } \frac{a}{c} < \frac{b}{5} \text{ and } \frac{b}{c} = \frac{5}{7}\]

Since \(a < b + c\),

\[\frac{a}{c} = \frac{b}{c} + 1 \Rightarrow \frac{h}{5} < \frac{h}{7} + 1\]
\[h \cdot \left(\frac{1}{5} - \frac{1}{7}\right) < 1\]

so \(h < \frac{7.5}{2.5} = 17.5\)

4. Answer: \(a^6 - 6a^4b + 9a^2b^2 - 2b^3\)

Note: \((x^{n-1} + y^{n-1})(x + y) = x^n + y^n + xy^{n-1} + xy^{n-1} = x^n + y^n + xy(x^{n-2} + y^{n-2})\). Thus, let \(f(n) = x^n + y^n\). We see \(f(n) = af(a - 1) - bf(n - 2)\).

\[x^0 + y^0 = 2, \text{ so } f(0) = 2\]
\[x^1 + y^1 = x + y = a, \text{ so } f(1) = a\]
\[f(2) = a^2 - 2b\]
\[f(3) = a^3 - 3ab\]
\[f(4) = a^4 - 3a^2b - a^2b + 2b^2 = a^4 - 4a^2b + 2b^2\]
\[f(5) = a^5 - 4a^3b + 2ab^2 - a^3b + 3ab^2 = a^5 - 6a^3b + 5ab^2\]
\[f(6) = a^6 - 5a^4b + 5a^2b^2 - a^4b + 4a^2b^2 - 2b^3 = a^6 - 6a^4b + 9a^2b^2 - 2b^3\]
5. Answer: 1

\[
\sin(\arccos(\tan(\arcsin x))) = x \\
\sin \left( \arccos \left( \frac{x}{\sqrt{1-x^2}} \right) \right) = x \\
\sqrt{1 - \left( \frac{x}{\sqrt{1-x^2}} \right)^2} = x \\
\sqrt{1 - 2x^2} = x \\
1 - 2x^2 = x^2 - x^4 \\
x^4 - 3x^2 + 1 = 0
\]

Solving and restricting \(x\) to positive numbers:

\(x^2 = \frac{3 + \sqrt{5}}{2}\) or \(x = \sqrt{\frac{3 - \sqrt{5}}{2}}\). Multiplying these together, the answer is \(\sqrt{\frac{9 - \sqrt{5}}{4}}\).

6. Answer: 8024

Write the expression as \(x^4 + x^2 + 1\) where \(x = 2^n\). This is equivalent to \((x^2 + 1)^2 - x^2\) (by adding and subtracting \(x^2\)). This expression can be written as \((x^2 + x + 1)(x^2 - x + 1) = \frac{x^n - 1}{x - 1} \cdot \frac{x^{n+1} - 1}{x^2 - 1} = \frac{2^{n+1} - 1}{2^n - 1}\). Hence \(p(n) = 6n\) and \(q(n) = 2n\). It’s not hard to see that this is the only solution by considering the limit of each expression as \(n\) approaches infinity. The highest-order terms predominate: \(2^{4n}\) and \(2^{q(n)p(n)/q(n)(q(n)-1)}\). This implies that \(p\) and \(q\) are linear functions. Exact functions can be determined by evaluating the expressions at \(n = 1\) and \(n = 2\) and solving for two variables. The answer is 8,024.

7. Answer: \(\frac{1}{12}\)

This is a geometric probability problem. The set of 3-tuples above fits an equilateral triangle on the plane \(x + y + z = 3000\). We’re going to look at the sections of this triangle where \(x \geq 2500\). This is a triangle with vertices \((2500, 0, 500)\), \((2500, 0, 500)\), and \((3000, 0, 0)\). This is an equilateral triangle with length \(500\sqrt{2}\). The area of this triangle is \(\frac{\text{side}^2\sqrt{3}}{4} = \frac{(300\sqrt{2})^2\sqrt{3}}{4} = 125000\sqrt{3}\). Since \(x\), \(y\), or \(z\) can be larger than 2500, we need to multiply this by 3 to get the total area that works: \(125000\sqrt{3} \cdot 3 = 375000\sqrt{3}\). The total possible area is the whole triangle of side length \(3000\sqrt{2}\): \(\frac{\text{side}^2\sqrt{3}}{4} = \frac{(3000\sqrt{2})^2\sqrt{3}}{4} = 450000\sqrt{3}\). So the overall probability is \(\frac{375000\sqrt{3}}{4500000\sqrt{3}} = \frac{1}{12}\).
8. Answer: 64
Let \( S_n = \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}} \)

\[
\sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{(n+1)^2}} < S_n < \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{n^2}}
\]

\[
((n+1)^2 - n^2 + 1) \frac{1}{n+1} < S_n < ((n+1)^2 - n^2 + 1) \frac{1}{n}
\]

\[
\frac{2(n+1)}{n+1} < S_n < \frac{2(n+1)}{n}
\]

\[
2 < S_n < 2 + \frac{1}{n}
\]

Thus \( \lim_{n \to \infty} \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}} = 2 \)

9. Answer: \( \frac{5}{2} \)
Suppose the medians intersect at \( P \). If \( BC = x, BP = CP = \frac{x}{2\sqrt{2}} \). By a well-known property of centroids, \( \frac{MP}{MB} = \frac{1}{3} \), so \( MP = \frac{x}{2\sqrt{2}} \). Using the Pythagorean Theorem, we find that \( MB = \frac{x\sqrt{2}}{2} \) and so \( AB = x\sqrt{\frac{5}{2}} \). So \( \left(\frac{AB}{BC}\right)^2 = \frac{5}{2} \).

10. Answer: 638
Notice that \( n^2 + 8 \) is divisible by \( n + 2 \). Therefore, \( m - 8 \) must be divisible by \( n + 2 \) for the expression to be an integer. If \( f \) is a factor of \( m - 8 \), \( n = f - 2 \) is a corresponding suitable \( n \); we then need \( f \geq 3 \) to make \( n > 0 \). Thus \( m - 8 \) must have twelve each odd and even factors including 1 and 2. To make the number of odd and even factors equal in order to minimize \( m \), the power of 2 in the prime factorization of \( m - 8 \) must be 1. Suppose the prime factorization of \( m - 8 \) is then \( 2^4 \cdot 3^a \cdot 5^b \cdot 7^c \cdot 11^d \) (larger prime factors will clearly not minimize \( m \)). Then \( (a+1)(b+1)(c+1)(d+1) \geq 12 \). To minimize \( m \), \( a \geq b \geq c \geq d \). We then examine values of \( \frac{m-8}{2} \) to determine the best \( (a, b, c, d) \). \( 3 \cdot 5 \cdot 7 \cdot 11 = 1155 \), \( 3^2 \cdot 5 \cdot 7 = 315 \). Moving any more factors into smaller primes involves multiplying by \( \frac{2}{3} \) or \( \frac{2}{5} \) (or subsequent larger powers of 3), which increases the value. Therefore \( m - 8 = 2 \cdot 3^2 \cdot 5 \cdot 7 \), so \( m = 638 \).

11. Answer: 64
Using the first condition with \( j = 1003 \) we get \( c_i = 2(1003 - i)c_{2006-i} \). Replace the coefficients of \( P \) in this manner and notice that \( x^{2006} P \left( \frac{2}{2006} \right) = P(x) \). Therefore if \( r \) is a solution of \( P(x) = 0 \) then \( P(2/r) = 0 \). Then:

\[
\sum_{i \neq j, i=1,j=1}^{2006} \frac{r_i}{r_j} = \sum_{i=1}^{2006} r_i \sum_{i=1}^{2006} \frac{1}{r_i} - 2006 = \frac{1}{2} \left( \sum_{i=1}^{2006} r_i \right)^2 - 2006 = 42
\]

Solving for the desired sum gives 64.

12. Answer: 17
\[
\sum_{i=1}^{k} \left( 180 - \frac{360}{n_i} \right) = 0, \text{ so } k/2 - 1 = \sum_{i=1}^{k} \frac{1}{n_i}. \text{ Clearly, } 3 \leq k \leq 6, \text{ since the interior angles are less than } 180^\circ, \text{ and six equilateral triangles maximize } k. \text{ For each } k, \text{ bounds can be established on the smallest or largest } n_i. \text{ From then, we can fix all but two of the } n_i, \text{ solve algebraically, then use reasonable guesswork to find all integer solutions. For } k = 3, \text{ fix } n_1 \text{ at } 3, 4, 5, \text{ or } 6 \text{ and then solve } \frac{3}{2} - 1 = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}. \text{ This yields 10 solutions. For } k = 4, n_4 = 3 \text{ or } 4; \text{ there are 4 solutions. For } k = 5, n_5 = n_4 = n_3 = 3, \text{ giving two solutions. Finally there is of course only one solution for } k = 6. \text{ } 10 + 4 + 2 + 1 = 17
\]
13. Answer: \( \frac{2\sqrt{7}}{7} \)

It is clear from drawing the graph that we want to find the cosine of the smallest angle \( \theta (0 < \theta < \frac{\pi}{2}) \) such that a ray leaving the origin at angle \( \theta \) will hit the graph of the hyperbola in the first quadrant. Since \( \cos \theta \) is a decreasing function on this interval, we want the largest possible value of \( \cos \theta \).

We begin by writing the hyperbola in polar coordinates: \( r^2 \sin^2 \theta = r^2 \cos^2 \theta - r \cos \theta + 1. \)

Using \( \sin^2 \theta = 1 - \cos^2 \theta \) and collecting like terms, we get: \( (2 \cos^2 \theta - 1)r^2 - (\cos \theta)r + 1 = 0. \)

Now we can use the quadratic formula to solve for \( r \):

\[
r = \frac{\cos \theta \pm \sqrt{\cos^2 \theta - 4(2 \cos^2 \theta - 1)}}{4 \cos^2 \theta - 2}
\]

If there are any solutions for \( r \), the quantity under the square root must be nonnegative:

\[
\cos^2 \theta \geq 8 \cos^2 \theta - 4
\]

\[
7 \cos^2 \theta \leq 4
\]

\[
\cos \theta \leq \frac{2\sqrt{7}}{7}
\]

So the angle we are looking for has

\[
\cos \theta = \frac{2\sqrt{7}}{7}
\]

14. Answer: 292

First we find the largest power of an integer \( d \) that divides \( k \). Notice that \( \left\lfloor \frac{k}{7} \right\rfloor \) of the integers 1, 2, \ldots, \( k \) are divisible by \( d \), \( \left\lfloor \frac{k}{d^2} \right\rfloor \) are divisible by \( d^2 \), and so on. The largest power we are looking for is then \( \left\lfloor \frac{k}{7} \right\rfloor + \left\lfloor \frac{k}{7^2} \right\rfloor + \left\lfloor \frac{k}{7^3} \right\rfloor + \ldots \). Now let \( m = 2006 - n \), so that \( \binom{2006}{n} = \frac{2006!}{n!(m^n)} \); the largest power of 7 divisor is then \( \left( \binom{2006}{m} - \left\lfloor \frac{m}{7} \right\rfloor + \left\lfloor \frac{m}{7^2} \right\rfloor - \left\lfloor \frac{m}{7^3} \right\rfloor + \ldots \right) + \left( \binom{2006}{m} - \left\lfloor \frac{m}{7} \right\rfloor + \left\lfloor \frac{m}{7^2} \right\rfloor - \left\lfloor \frac{m}{7^3} \right\rfloor + \ldots \right) + \ldots \). Note that if \( \frac{m}{7} = \left\lfloor \frac{m}{7} \right\rfloor + n' \) and \( \frac{m}{7^2} = \left\lfloor \frac{m}{7^2} \right\rfloor + m' \), then \( \frac{2006}{d} = \frac{n + m}{7^2} \) leaves a remainder of \( r = n' + m' - d \), whichever satisfies \( 0 \leq r < d \). Therefore \( \left\lfloor \frac{2006}{d} \right\rfloor - \left\lfloor \frac{m}{7} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor = 0 \). To make this 1 in order to get large divisors of \( \binom{2006}{n} \), we need \( m', n' > r \). We therefore find the remainders when 2006 is divided by 7, \( 7^2 \), and \( 7^3 \): 4, 46, and 291. Therefore \( n \) must leave a remainder of at least 292 when divided by 343, so we try \( n = 292 \), which has remainders of 5 and 47 when divided by 7 and 49.

15. Answer: \( \frac{12}{\pi^2} \)

Write

\[
\prod_{p \text{ prime}} \frac{p^2}{p^2 - 1} \prod_{c \text{ composite}} \frac{c^2}{c^2 - 1} = \prod_{n=2}^{\infty} \frac{n^2}{n^2 - 1} = \prod_{n=2}^{\infty} \frac{n}{n - 1} \frac{n}{n + 1}
\]

which telescopes and evaluates to 2. Meanwhile we can write

\[
\prod_{p \text{ prime}} \frac{p^2}{p^2 - 1} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^2}}.
\]

The latter is equivalently rewritten:

\[
\prod_{p \text{ prime}} 1 + \frac{1}{p^2} + \frac{1}{p^4} + \ldots = \prod_{p \text{ prime}} \left( \sum_{n=0}^{\infty} \frac{1}{p^{2n}} \right).
\]

When we distribute the infinite product over the infinite sum, we get a sum of terms. Each term is of the form \( \frac{1}{m^2} \) for integer \( m \). Each \( m \) appears exactly once, so the product is equal to \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).

Hence

\[
\prod_{c \text{ composite}} \frac{c^2}{c^2 - 1} = \frac{2}{\pi^2} \frac{12}{\pi^2} = \frac{12}{\pi^2}.
\]