1. Given $\triangle ABC$, where $A$ is at $(0,0)$, $B$ is at $(20,0)$, and $C$ is on the positive $y$-axis. Cone $M$ is formed when $\triangle ABC$ is rotated about the $x$-axis, and cone $N$ is formed when $\triangle ABC$ is rotated about the $y$-axis. If the volume of cone $M$ minus the volume of cone $N$ is $140\pi$, find the length of $BC$.

2. In a given sequence $\{S_1, S_2, \ldots, S_k\}$, for terms $n \geq 3, S_n = \sum_{i=1}^{n-1} i \cdot S_{n-i}$. For example, if the first two elements are 2 and 3, respectively, the third entry would be $1 \cdot 3 + 2 \cdot 2 = 7$, and the fourth would be $1 \cdot 7 + 2 \cdot 3 + 3 \cdot 2 = 19$, and so on. Given that a sequence of integers having this form starts with 2, and the 7th element is 68, what is the second element?

3. A triangle has altitudes of lengths 5 and 7. What is the maximum length of the third altitude?

4. Let $x + y = a$ and $xy = b$. The expression $x^6 + y^6$ can be written as a polynomial in terms of $a$ and $b$. What is this polynomial?

5. There exist two positive numbers $x$ such that $\sin(\arccos(\tan(\arcsin x))) = x$. Find the product of the two possible $x$.

6. The expression $16^n + 4^n + 1$ is equivalent to the expression $(2^{p(n)} - 1)/(2^{q(n)} - 1)$ for all positive integers $n > 1$ where $p(n)$ and $q(n)$ are functions and $\frac{p(n)}{q(n)}$ is constant. Find $p(2006) - q(2006)$.

7. Let $S$ be the set of all 3-tuples $(a,b,c)$ that satisfy $a + b + c = 3000$ and $a,b,c > 0$. If one of these 3-tuples is chosen at random, what’s the probability that $a$, $b$, or $c$ is greater than or equal to 2,500?

8. Evaluate: \( \lim_{n \to \infty} \sum_{k=n^2}^{(n+1)^2} \frac{1}{\sqrt{k}} \)

9. $\triangle ABC$ has $AB = AC$. Points $M$ and $N$ are midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. The medians $\overline{MC}$ and $\overline{NB}$ intersect at a right angle. Find $(\frac{\overline{MC}}{\overline{NB}})^2$.

10. Find the smallest positive integer $m$ for which there are at least 11 even and 11 odd positive integers $n$ so that $n^3 + m$ is an integer.

11. Polynomial $P(x) = c_{2006}x^{2006} + c_{2005}x^{2005} + \ldots + c_1x + c_0$ has roots $r_1, r_2, \ldots, r_{2006}$. The coefficients satisfy $2i\frac{c_i}{c_{2006-i}} = 2j\frac{c_j}{c_{2006-j}}$ for all pairs of integers $0 \leq i, j \leq 2006$. Given that $\sum_{i=1}^{2006} \frac{c_i}{r_j} = 42$, determine $\sum_{i=1}^{2006} (r_1 + r_2 + \ldots + r_{2006})$.

12. Find the total number of $k$-tuples $(n_1, n_2, \ldots, n_k)$ of positive integers so that $n_{i+1} \geq n_i$ for each $i$, and $k$ regular polygons with numbers of sides $n_1, n_2, \ldots, n_k$ respectively will fit into a tessellation at a point. That is, the sum of one interior angle from each of the polygons is $360^\circ$.

13. A ray is drawn from the origin tangent to the graph of the upper part of the hyperbola $y^2 = x^2 - x + 1$ in the first quadrant. This ray makes an angle of $\theta$ with the positive $x$-axis. Compute $\cos \theta$.

14. Find the smallest nonnegative integer $n$ for which $\binom{2006}{n}$ is divisible by $7^3$.
15. Let $c_i$ denote the $i$th composite integer so that \{c_i\} = 4, 6, 8, 9, .... Compute

$$\prod_{i=1}^{\infty} \frac{c_i^2}{c_i^2 - 1}.$$ 

(Hint: $\sum_{i=1}^{n} \frac{1}{n^2} = \frac{\pi^2}{6}$).