CALCULUS TEST SOLUTIONS
STANFORD MATH TOURNAMENT
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1. Evaluate \( \int_{0}^{8} \frac{d}{dx}(4x^2 + 3x + 2) \, dx \).
   **Answer:** 280

   Solution:
   \[
   \int_{0}^{8} \frac{d}{dx}(4x^2 + 3x + 2) \, dx = \int_{0}^{8} (8x + 3) \, dx \\
   = 4x^2 + 3x \bigg|_{0}^{8} \\
   = 4 \cdot 8^2 + 3 \cdot 8 \\
   = 280
   \]

2. From atop a very tall building, a spherical cow drops a lit firecracker. He hears it explode 10 seconds later. Given that sounds travels at about \( \frac{300}{s} \) and acceleration due to gravity is \( \frac{10}{s^2} \) Find the length of time it falls before it exploded. Remember, that acceleration is change in velocity over time and that the distance covered by a falling object is the area under the velocity curve.
   **Answer:** \(-30 + 10\sqrt{15}\)

   Solution: Let \( t_0 \) be the amount of time the object spent falling. The distance it travels in this time is \( 5t_0^2 \). The sound travels for \( 10 - t_0 \) seconds, and it goes the same distance up as the object fell. Thus, \( 300(10 - t) = 5t^2 \), which we can solve using the quadratic equation, obtaining \(-30 + 10\sqrt{15}\).

3. A spherical balloon expands at a rate of 1 cm\(^3\)/sec. In terms of the radius \( r \), at what rate is the radius of the balloon increasing?
   **Answer:** \( \frac{1}{4\pi r^2} \) cm/sec

   Solution: As a function of the radius, the volume \( V(r) \) of the balloon is given by \( V(r) = \frac{4}{3}\pi r^3 \), so \( \frac{dV}{dr} = 4\pi r^2 \). Meanwhile, the chain rule tells us that \( \frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} \). And we are given \( \frac{dV}{dt} = 1 \), so \( \frac{dr}{dt} = \left( \frac{dV}{dr} \right)^{-1} = \frac{1}{4\pi r^2} \).

4. Consider an isosceles trapezoid with vertices at \((0,0),(2,3),(6,3),(8,0)\), and let \( d_1 \) and \( d_2 \) be the diagonals. If we let \( f(t) \) be the distance between \( d_1 \) and \( d_2 \) along the line \( y = t \), then what is the average value of \( f(t) \) over the interval \([0,3]\) (the whole trapezoid)?
   **Answer:** \( \frac{10}{3} \)

   Solution: The average value of \( f(y) \) can be graphically represented as the sum of the areas of the two triangles formed between the intersection of \( d_1 \) and \( d_2 \) and the two bases of the trapezoid divided by the range of the interval. So, since the intersection of \( d_1 \) and \( d_2 \) is at \((4,2)\), the area of the top triangle is 2 and the area of the bottom triangle is 8, and the total area is 10, so the average value of \( f(y) \) is \( \frac{10}{3} \).

5. Suppose \( f(x) = e^{ax} + e^{bx} \), where \( a \neq b \), and that \( f'' - 2f' - 15 = 0 \) for all \( x \). Give all possible ordered pairs \((a,b)\).
   **Answer:** \((3, -5)\) or \((-5, 3)\)

   Solution: First, we calculate \( f'(x) = ae^{ax} + be^{bx} \) and \( f''(x) = a^2e^{ax} + b^2e^{bx} \). Substituting these into the equation we are given and grouping terms, we obtain \((a^2 - 2a - 15)e^{ax} + (b^2 - 2b - 15)e^{bx} = 0\). This can only be zero for all \( x \) if the coefficient of each exponential is zero, so 3 and \(-5\) are the only possible values for both \( a \) and \( b \). The condition \( a \neq b \) makes \((3, -5)\) and \((-5, 3)\) the only possible answers.
6. Evaluate $\int_{-1/2}^{1/2} \sqrt{1-x^2} \, dx$.

**Answer:** $\pi/6 + \sqrt{3}/4$

Solution: Draw the unit circle. The area to be evaluated is the region between $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, which is composed of one-sixth of the unit circle and two triangles with base $1/2$ and height $\sqrt{3}/2$. The total area is then clearly $\pi/6 + \sqrt{3}/4$.

7. What is the derivative of $x^e$, assuming $x > 0$?

**Answer:** $x^e (\ln x + 1)$

Solution: First, we write $x^e = e^{\ln x} = e^x \ln x$. Differentiating this expression via the chain rule, we obtain $\frac{d}{dx}(e^{\ln x}) = e^x \ln x (\ln x + 1) = x^e (\ln x + 1)$.

8. Evaluate

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2 + t^3}{1 + \sin t} \, dt.$$

**Answer:** $1/3$

Solution: Applying L'Hospital's rule, we see that

$$\lim_{x \to 0} \frac{\int_0^x \frac{t^2 + t^3}{1 + \sin t} \, dt}{x^3} = \lim_{x \to 0} \frac{x^2 + x^3}{3x^2} \sin x = \lim_{x \to 0} \frac{1 + x}{3(1 + \sin x)} = \frac{1}{3}.$$

9. Ford has an ivory bathtub in the shape of a cone, apex down, filled with sand. The cone has height 10 ft and maximum radius of 8 ft. He pulls the plug out of the apex and the sand drains out such that the radius decreases at a constant rate of $\frac{5}{16}$ ft/min. The sand drains into a paraboloid shaped swimming pool with a horizontal cross-section of a circle and a vertical cross-section which follows the graph of $h = r^2$, and it is already filled up to a height of 6 ft with water. Assuming the sand sinks quickly, at what rate is the water in the swimming pool rising when the height is 8 ft?

**Answer:** $\frac{1}{16} 5^{1/3} \cdot 299^{2/3}$ ft/min

Solution: Denote the volume of sand in the bathtub by $V_b$, its height by $h_b$, and the radius of the circle at the top of the sand by $r_b$. Then $V_b = \frac{1}{3} \pi r_b^2 h_b = \frac{5}{16} \pi r_b^3$ since $h_b = \frac{5}{8} r_b$ by similar triangles. Therefore, using the notation $f' = \frac{df}{dt}$ for functions $f$, we have $V'_b = \frac{5}{16} \pi r_b^2 r'_b = \frac{5}{8} \pi r_b^3$ since we are given $r'_b = 0.5$.

Now, let $V_p$ denote the volume of water in the pool, and let $h_p$ denote the height of the water. Using volume of rotation, $V_p = \frac{1}{3} \pi h_p^2$, and $V'_p = \pi h_b h'_b$ (keep in mind we want to solve for $h'_b$). When $h_b = 8$, we have $V_p = 32 \pi$, and $18 \pi$ of that is water, so $14 \pi$ is sand. Thus, $V_b = \frac{640 \pi}{3} - 14 \pi = \frac{598 \pi}{3}$, and $r_b = 2 \sqrt{\frac{299}{6}}$. The bathtub drains at the same rate the pool fills, so $V'_p = V'_b$, which gives us

$$\pi h_b h'_b = \frac{5}{8} \pi r_b^2$$

$$8h'_b = \frac{1}{2} 5^{1/3} \cdot 299^{2/3}$$

$$h'_b = \frac{1}{16} 5^{1/3} \cdot 299^{2/3}.$$

10. What is $e^{1/2} \cdot e^{-1/2} \cdot e^{1/3} \cdot e^{-1/4} \ldots$ ?

**Answer:** 2
Solution: Let $P = e^1 \cdot e^{-1/2} \cdot e^{1/3} \cdot e^{-1/4} \cdots$. Then

$$
\ln P = \ln e + \ln e^{-1/2} + \ln e^{1/3} + \ln e^{-1/4} + \cdots \\
= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots
$$

Now recall from the study of power series that for $|x| < 1$,

$$
\ln (x + 1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n.
$$

This formula works for $x = 1$ as well, and using it, we obtain $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. Hence, $\ln P = \ln 2$, so $P = 2$. 
