2001 Stanford Math Tournament
Advanced Topics

1. Suppose $\sinh(x) = \frac{e^x - e^{-x}}{2}$. What is the inverse function of $\sinh(x)$?

   Solution: $\mathbf{\ln(x + \sqrt{x^2+1})}$

   $y = \frac{e^x - e^{-x}}{2}$. Switch x and y to find the inverse, so $x = \frac{e^y - e^{-y}}{2}$, or $2x = e^y - e^{-y}$, so $2xe^y = e^{2y} - 1$, and $e^{2y} - 2xe^y - 1 = 0$. By using the quadratic formula, we see that $e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$, so $y = \ln(x + \sqrt{x^2+1})$ (since y cannot be $\ln(x - \sqrt{x^2+1})$, since $x - \sqrt{x^2+1}$ is always negative, and ln is not defined for negative numbers).

2. Write $20_{10}\frac{70_{17}!}{170_8!}$ as a decimal in base 6. The subscript indicates the base in which the number is written (i.e., 20_{10} is 20 base 10.)

   Solution: $\mathbf{0.1_6}$ **or 0.1**

   The easiest way to solve this problem is to convert everything into base 10.

$$
\begin{aligned}
70_{17} &= 7 \times 17^1 + 0 \times 17^0 = 119 \\
170_8 &= 1 \times 8^2 + 7 \times 8^1 + 0 \times 8^0 = 120 \\
20_{10}\frac{70_{17}!}{170_8!} &= 20\frac{119!}{120!} = \frac{1}{6}
\end{aligned}
$$

   Converting to base 6, $\frac{1}{6} = 0.1_6$.

3. There are 36 penguins in a row, and Barbara Manatee is standing in front of them. In general, a penguin rotation of penguins $p_1, p_2, \ldots, p_n$ is a rearrangement of them such that p_1 moves to where p_2 was standing, and in general p_i moves to where p_{i+1} was standing, and p_n moves to where p_1 was standing. So, after a penguin rotation, the new order of these penguins is $p_n, p_1, p_2, \ldots, p_{n-1}$. Whenever Barbara Manatee blows her whistle, the 2-4 penguins go through a penguin rotation, the 5-9 penguins go through a penguin rotation, the 10-16 penguins go through a penguin rotation, the 17-25 penguins go through a penguin rotation, and the 26-36 penguins go through a penguin rotation. What is the least positive number of whistle blows such that the penguins all return to their original position?

   Solution: 3465

   Since we have groups of 3,5,7,9,11 penguins rotating, for them all to be in the right place requires the number of whistle blows to be a multiple of the least common multiple of 3,5,7,9,11 (denoted $[3,5,7,9,11]$). So, we calculate $[3,5,7,9,11] = [5,7,9,11] = 5 \cdot 7 \cdot 9 \cdot 11 = \mathbf{3465}$.

4. Eleven pirates find a treasure chest. When they split up the coins in it, evenly among all the pirates, they find that there are 5 coins left. They throw one pirate overboard and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split perfectly. What is the least number of coins there could have been?

   Solution: 423

   Let n be the number of coins in the chest. Then, $n \equiv 5 \pmod{11}$ (this means that the remainder of n when divided by 11 is 5), $n \equiv 3 \pmod{10}$, and $n \equiv 0 \pmod{9}$. The Chinese Remainder Theorem gives an elegant solution to this problem, or you can solve it as follows. Since $n \equiv 3 \pmod{10}$, we know that the last digit of n must be 3. Also, since $n \equiv 0 \pmod{9}$, n is divisible by 9, so $n = 9q$ for some q. Since n ends in a 3, q must end in a 7. That means that q looks like $10p+7$, so $n = 9(10p+7) = 90p + 63$. Now, you can just try $p = 1, 2, \ldots$, and you'll find that $p = 4$ works.

5. Evaluate $1 \cdot 2^{-1} + 3 \cdot 2^{-2} + 5 \cdot 2^{-3} + 7 \cdot 2^{-4} + \ldots$

   Solution: 3

We will express this as a sum of geometric series as follows:

$$\begin{aligned}
1 \cdot 2^{-1} + 3 \cdot 2^{-2} + 5 \cdot 2^{-3} + 7 \cdot 2^{-4} + \ldots &= \sum_{n=1}^{\infty} (2n-1)2^{-n} \\
&= \sum_{n=1}^{\infty} (2n)2^{-n} - \sum_{n=1}^{\infty} 2^{-n} \\
&= \sum_{n=1}^{\infty} n2^{-n+1} - \frac{1/2}{1-1/2} \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{n} 2^{-n+1} - 1 \\
&= \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} 2^{-n+1} - 1 \\
&= \sum_{j=1}^{\infty} \frac{2^{-j+1}}{1-1/2} - 1 \\
&= \sum_{j=1}^{\infty} 2^{-j+2} - 1 \\
&= \frac{2}{1-1/2} - 1 \\
&= 3
\end{aligned}$$

6. How many subsets of $\{n \mid n > 0$ and n is a multiple of 3 less than $100\}$ are also subsets of $\{n \mid n > 0$ and n is a multiple of 4 less than $100\}$?

   Solution: 256

   Let $A = \{n \mid n > 0$ and n is a multiple of 3 less than $100\}$ and $B = \{n \mid n > 0$ and n is a multiple of 4 less than $100\}$. Then, $A \cap B$ consists of all multiples of 12 less than 100; there are 8 of these. The subsets of A that are also subsets of B are exactly the subsets of $A \cap B$; since $A \cap B$ has 8 elements, it has $2^8 = 256$ subsets.

7. There are 2000 dots spaced evenly around a circle. If 4 distinct dots A, B, C, and D are picked randomly, what is the probability that AB intersects CD?

   Solution: $\frac{1}{3}$

   Randomly pick the 4 dots A, B, C, and D. Now, if we start from A and go clockwise, there are 6 equally likely possibilities for the order that the other 3 dots will appear:

$$\begin{array}{ccc} B, C, D & C, B, D & D, B, C \\ B, D, C & C, D, B & D, C, B \end{array}$$

   AB and CD intersect if and only if B is the second of these three dots; this happens with probability $\frac{1}{3}$.

8. Ashley, Bob, Carol, and Doug are rescued from a desert island by a pirate who forces them to play a game. Each of the four, in alphabetical order by their first names, is forced to roll two dice. If the total on the two dice is either 8 or 9, the person rolling the dice is forced to walk the plank. The players go in order until one player loses: A, B, C, D, A, B, ... What is the probability that Doug survives?

   Solution: 148/175

We will instead find the probability that Doug doesn't survive. Notice that the only time Doug doesn't survive is if all four survive the first n rounds (where a round is once through all four), and then Alice, Bob, and Carol survive the $n+1$-st round but Doug does not. The probability of rolling an 8 or 9 is $\frac{5}{36}+\frac{4}{36}=\frac{1}{4}$, so the probability that all four survive a round is $\left(\frac{3}{4}\right)^4=\frac{81}{256}$. So, all four survive the first n rounds with probability $\left(\frac{81}{256}\right)^n$. The probability that Alice, Bob, and Carol survive the $n+1$-st round but Doug does not is $\left(\frac{3}{4}\right)^3\frac{1}{4}=\frac{27}{256}$. So, the probability that Doug has to walk the plank in the $n+1$-st round is $\frac{27}{256}\left(\frac{81}{256}\right)^n$. Thus, the probability that Doug doesn't survive is

$$\sum_{n=0}^{\infty}\frac{27}{256}\left(\frac{81}{256}\right)^n=\frac{27}{256}\frac{1}{1-\frac{81}{256}}=\frac{27}{175}.$$

So, the probability that Doug survives is $\frac{148}{175}$.

9. There are 19 men numbered 1 though 19 and 20 women numbered 1 through 20 entered in a computer dating service. The computer wants to match every man to a compatible woman, and each man is only compatible with women who have a number that is greater than equal to his (i.e. man 19 is only compatible with women 19 and 20, man 18 is only compatible with women 18, 19, 20, etc.). If each women is matched with at most one man, let n be the number of ways that the computer can match them. What is the prime factorization of n?

   Solution: $\mathbf{2^{19}}$

   There are 2 ways to match man 19, with either woman 19 or woman 20. Given that, there are 2 ways to match man 18, with either woman 18 or with the one that man 19 didn't choose. This continues all the way back to man 1, so there are 2 choices for each of the 19 men, so the total number of matchings is 2^{19}.

10. David is playing with Legos with velcro attached to the ends. He has green Legos of length 1, blue Legos of length 2, and red Legos of length 3, and wants to combine them (by attaching them at the ends) to make a "super-lego" of length 10. If any different ordering of colors is considered a distinct "super-lego", how many ways can he make this "super-lego"?

    Solution: 274

    Let a_n be the number of ways to make a super-lego of length n. Then a_1 is of course 1, $a_2=2$ (we can have either a lego of length 2 or 2 legos of length 1), and $a_3=4$. Now, observe that to get a super-lego of length n, we must have either taken a super-lego of length $n-3$ and attached a lego of length 3, a super-lego of length $n-2$ and attached a lego of length 2, or a super-lego of length $n-1$ and attached a lego of length 1. So, $a_n=a_{n-3}+a_{n-2}+a_{n-1}$. Using this relationship, it's easy to find that $a_{10}=274$.

2001 Stanford Math Tournament
Algebra

1. Find the result of adding seven to the result of forty divided by one-half.

   Solution: 87

   Since we are dividing forty by one-half, the result is 80 (not 20), and adding 7 gives 87.

2. Each valve A, B, and C, when open, releases water into a tank at its own constant rate. With all three valves open, the tank fills in 1 hour, with only valves A and C open it takes 1.5 hours, and with only valves B and C open it takes 2 hours. How many hours will it take to fill the tank with only valves A and B open?

   Solution: $1.2 = \frac{6}{5}$

   Let r_A, r_B, r_C be the rates of valves A, B, and C in tanks per hour, respectively. Then, $1(r_A+r_B+r_C) = 1$, $1.5(r_A + r_C) = 1$, and $2(r_B + r_C) = 1$. So, $r_A + r_B + r_C = 1$, $r_A + r_C = \frac{2}{3}$, and $r_B + r_C = \frac{1}{2}$. So, $r_A + r_B = 2(r_A + r_B + r_C) - (r_A + r_C) - (r_B + r_C) = 2 - \frac{2}{3} - \frac{1}{2} = \frac{5}{6}$. Thus, it takes $\frac{6}{5} = 1.2$ hours to fill the tank with only valves A and B open.

3. Julie has a 12 foot by 20 foot garden. She wants to put fencing around it to keep out the neighbor's dog. Normal fenceposts cost \$2 each while strong ones cost \$3 each. If Julie needs one fencepost for every 2 feet and has \$70 to spend on fenceposts, what is the greatest number of strong fenceposts she can buy?

   Solution: 6

   Let x be the number of normal fenceposts and y be the number of strong fenceposts. The perimeter of Julie's garden is 64 feet, and she needs one fencepost every 2 feet, so $x + y = 32$. The total cost of the fenceposts is $2x + 3y$, and we want $2x + 3y \leq 70$. Since $2x + 2y = 64$, we find that $y \leq 6$.

4. $p(x)$ is a real polynomial of degree at most 3. Suppose there are four distinct solutions to the equation $p(x) = 7$. What is $p(0)$?

   Solution: 7

   Since the degree of $p(x) - 7$ is at most 3, it has at most 3 roots. But we have 4 solutions, so $p(x) - 7$ must be constant and equal to zero. So, $p(x) = 7$, and thus $p(0) = 7$.

5. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = \begin{cases} 2, & x = 0 \\ (f(x-1))^2, & x \neq 0 \end{cases}$ What is $\log_2 f(11)$?

   Solution: 2048

   For $x \neq 0$, $\log_2 f(x) = 2\log_2 f(x-1)$. For $x = 0$, $\log_2 f(x) = 1$. So, in general, $\log_2 f(x) = 2^x$. Thus, $\log_2 f(11) = 2^{11} = 2048$.

6. If for three distinct positive numbers x, y, and z,

   $$\frac{y}{x-z} = \frac{x+y}{z} = \frac{x}{y},$$

   then find the numerical value of x/y.

   Solution: 2

   Since $\frac{y}{x-z} = \frac{x+y}{z}$, $yz = x^2 + xy - xz - yz$. Since $\frac{y}{x-z} = \frac{x}{y}$, $y^2 = x^2 - xz$. Finally, since $\frac{x+y}{z} = \frac{x}{y}$, $xz = yx + y^2 = yx + x^2 - xz = yz + yz = 2yz$. So, $x/y = 2$.

7. If $\log_A B + \log_B A = 3$ and $A < B$, find $\log_B A$.

   Solution: $\frac{3-\sqrt{5}}{2}$

   Let $x = \log_B A$. Then $B^x = A$, so $B = A^{\frac{1}{x}}$. Thus, $\log_A B = \frac{1}{x}$. So, $x + \frac{1}{x} = 3$. Multiplying through by x, $x^2 + 1 = 3x$, so $x^2 - 3x + 1 = 0$. By the quadratic formula, $x = \frac{3\pm\sqrt{5}}{2}$ Since $A < B$, $\log_B A < 1$, so $x = \frac{3-\sqrt{5}}{2}$

8. Determine the value of $1+\frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{1+\cdots}}}}$.

Solution: $\mathbf{\frac{1+\sqrt{3}}{2}}$

Let $x=1+\frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{1+\cdots}}}}$. Then, observe that $x=1+\frac{1}{2+\frac{1}{x}}$, so $x=1+\frac{x}{2x+1}=\frac{3x+1}{2x+1}$. Therefore, $2x^2+x=3x+1$, so $2x^2-2x-1=0$. By the quadratic formula, $x=\frac{2\pm\sqrt{4+8}}{4}=\frac{2\pm2\sqrt{3}}{4}=\frac{1\pm\sqrt{3}}{2}$. Now, $1+\frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{1+\cdots}}}}$ is clearly positive, so it must be $\frac{1+\sqrt{3}}{2}$.

9. Find all solutions to $(x-3)(x-1)(x+3)(x+5)=13$.

Solution: $\mathbf{\pm\sqrt{3}-1,\pm\sqrt{17}-1}$

Let $y=x+1$, so $x=y-1$, and our new equation is $(y-4)(y-2)(y+2)(y+4)=13$, or $(y^2-16)(y^2-4)=13$, and by expanding, we get that $y^4-20y^2+51=0$, so $(y^2-3)(y^2-17)=0$, so $y=\pm\sqrt{17},\pm\sqrt{3}$, and $x=\mathbf{\pm\sqrt{3}-1,\pm\sqrt{17}-1}$.

10. Suppose x,y,z satisfy

$$\begin{aligned} x+y+z &= 3 \\ x^2+y^2+z^2 &= 5 \\ x^3+y^3+z^3 &= 7 \end{aligned}$$

Find $x^4+y^4+z^4$.

Solution: 9

We have

$$9 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+xz) = 5+2(xy+yz+xz)$$

so $xy+yz+xz=2$. Similarly,

$$\begin{aligned} 15 &= (x^2+y^2+z^2)(x+y+z) \\ &= x^3+y^3+z^3+x^2y+y^2x+y^2z+z^2y+z^2x+x^2z \\ &= 7+xy(x+y)+yz(y+z)+zx(z+x) \\ &= 7+xy(3-z)+yz(3-x)+zx(3-y) \text{ [since } x+y+z=3] \\ &= 7+3(xy+yz+zx)-3xyz \\ &= 13-3xyz, \text{ since } xy+yz+zx=2 \end{aligned}$$

So, $xyz=\frac{-2}{3}$. Finally,

$$\begin{aligned} 21 &= (x^3+y^3+z^3)(x+y+z) \\ &= x^4+y^4+z^4+x^3y+y^3x+y^3z+z^3y+z^3x+x^3z \\ &= x^4+y^4+z^4+xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2) \\ &= x^4+y^4+z^4+xy(5-z^2)+yz(5-x^2)+zx(5-y^2) \\ &= x^4+y^4+z^4+5(xy+yz+zx)-xyz(x+y+z) \\ &= x^4+y^4+z^4+12 \end{aligned}$$

So, $x^4+y^4+z^4=9$.

2001 Stanford Math Tournament
Calculus

1. What is $\frac{d^{2959}}{dx^{2959}} \sin x$?

   Solution: $\boldsymbol{-\cos x}$

   Notice that the fourth derivative of $\sin x$ is again $\sin x$, and $2959 = 4 \times 739 + 3$. So, the 2959-th derivative of $\sin x$ is the same as the third derivative of $\sin x$, which is just $-\cos x$.

2. If $f(x) = [x]$ is the greatest integer function, what is $f'''(3.7)$?

   Solution: 0

   Notice that f is constant on the interval $(3, 4)$, so $f' = 0$ on $(3, 4)$. Thus, $f''' = 0$ on $(3, 4)$, so $f'''(3.7) = 0$.

3. Suppose that f is a monotonically increasing continuous function defined on the real numbers. We know that $f(0) = 0$ and $f(2) = 3$. Let S be the set of all possible values of $\int_0^2 f(x)\,dx$. What is the least upper bound of S?

   Solution: 6

   Since f is monotonically increasing, we know that for all $x \leq 2$, $f(x) \leq 3$. Also, for all $x \geq 0$, $f(x) \geq 0$. So, 6 is an upper bound of $\int_0^2 f(x)\ dx$. Now, we will show that it is the least upper bound. Consider functions f_n defined by

   $$f_n(x) = \begin{cases} 3nx, & x \leq \frac{1}{n} \\ 3, & x > \frac{1}{n} \end{cases}$$

   Then, $\int_0^2 f(x)\ dx = \int_0^{\frac{1}{n}} 3nx\ dx + \int_{\frac{1}{n}}^2 3\ dx = \frac{3}{2n} + 6 - \frac{3}{n} = 6 - \frac{3}{2n}$, so we have found monotonically increasing functions whose integrals are arbitrarily close to 6. Thus, the least upper bound of $\int_0^2 f(x)\ dx$ is 6.

4. Evaluate $\int_{-4}^{5} \frac{x^2}{|x|} dx$.

   Solution: $\boldsymbol{\frac{41}{2} = 20.5 = 20\frac{1}{2}}$

   For $x \leq 0, |x| = -x$. For $x \geq 0, |x| = x$. So,

   $$\begin{aligned} \int_{-4}^{5} \frac{x^2}{|x|}\, dx &= \int_{-4}^{0} \frac{x^2}{-x}\, dx + \int_0^5 \frac{x^2}{x}\, dx = \int_{-4}^{0} -x\, dx + \int_0^5 x\, dx \\ &= -\frac{1}{2}x^2\Big|_{-4}^{0} + \frac{1}{2}x^2\Big|_0^5 = -0 + \frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 25 - 0 \\ &= 8 + \frac{25}{2} \end{aligned}$$

   and this is $\boldsymbol{\frac{41}{2}}$.

5. $\int_1^\infty \frac{1}{x^2+3x} = ?$

   Solution: $\boldsymbol{\frac{1}{3}\ln 4 = \frac{\ln 4}{3}}$

   Solution: The integral given is equal to $\frac{1}{3}\lim_{u\to\infty} \int_1^u \left(\frac{1}{x} - \frac{1}{x+3}\right)$
   $= \frac{1}{3}\lim_{u\to\infty} (\ln u - (\ln(u + 3) - \ln 4))$
   $= \frac{1}{3}\ln 4.$

6. Given a point (p, q) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $p \neq 0$, find the x-intercept of the tangent line at (p, q) in terms of p, a, and b. (Note that $a, b \neq 0$.)

   Solution: $\boldsymbol{\frac{a^2}{p}}$

If $q = 0$, then we are either at the point $(a, 0)$ or $(-a, 0)$, so in either case, we see that $\frac{a^2}{p}$ gives the correct answer. Otherwise, suppose $q > 0$. Then, we are in the top half of the ellipse, which has equation $y = \sqrt{b^2 - \frac{x^2b^2}{a^2}}$. Remember that the slope of the tangent line at (p, q) is just $\frac{dy}{dx}$ at (p, q). By the chain rule, $\frac{dy}{dx} = -\frac{1}{2}\left(b^2 - \frac{x^2b^2}{a^2}\right)^{-1/2} \frac{2xb^2}{a^2} = -\frac{xb^2}{ya^2}$. So, at (p, q), $\frac{dy}{dx} = -\frac{pb^2}{qa^2}$. This means that the tangent line at (p, q) looks like $y = -\frac{pb^2}{qa^2}x + c$ for some c. We know that (p, q) is a point on the line, so $q = -\frac{pb^2}{qa^2}p + c$, and this tells us that $c = q + \frac{pb^2}{qa^2}p$. So, the equation of the tangent line is $y = -\frac{pb^2}{qa^2}x + q + \frac{pb^2}{qa^2}p$.

So, the x-intercept of the tangent line is the x satisfying $0 = -\frac{pb^2}{qa^2}x + q + \frac{pb^2}{qa^2}p$. Solving this linear equation, we find that $x = \frac{q^2a^2+p^2b^2}{pb^2}$. However, this is not yet in terms of p, a, and b. Remember that (p, q) is a point on the elliptic curve, so $\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1$. This means that $p^2b^2 + q^2a^2 = a^2b^2$; substituting this into the numerator of x, we get $x = \frac{a^2b^2}{pb^2} = \frac{a^2}{p}$.

Now, notice that the tangent line to $(p, -q)$ is just the tangent line of (p, q) reflected over the x-axis, so it has the same x-intercept.

7. Find the number of real solutions to $\sin(6\pi x) = x$, where x is in radians.

   Solution: 11

   There is one solution at $x = 0$. Now, consider the graph of $\sin(6\pi x) - x$. At $\frac{1}{12}$, $\sin(6\pi x) = \sin(\pi/2) = 1$, so $\sin(6\pi x) - x > 0$. At $\frac{3}{12}$, $\sin(6\pi x) - x < 0$, so there is a solution to $\sin(6\pi x) = x$ between $\frac{1}{12}$ and $\frac{3}{12}$. Similarly, there are solutions between $\frac{3}{12}$ and $\frac{5}{12}$, $\frac{5}{12}$ and $\frac{7}{12}$, $\frac{7}{12}$ and $\frac{9}{12}$, and $\frac{9}{12}$ and $\frac{11}{12}$, giving 5 solutions with x positive. There is not a solution between $\frac{11}{12}$ and 1, since $\sin(6\pi \cdot \frac{11}{12}) = -1$ and $\sin(6\pi) = 0$, and there are no solutions for $x > 1$, since then $x > 1 > \sin(6\pi x)$. Since the graph of $\sin(6\pi x) - x$ is symmetric around the origin, there must be 5 negative solutions as well, for a total of 11 solutions.

8. If a, b, and c are positive real numbers such that $a + b + c = 16$ and $a^2 + b^2 + c^2 = 160$, what is the maximum possible value of abc?

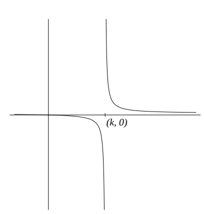
   Solution: $\frac{128}{27}(7\sqrt{7} - 10)$

   First, notice that $a+b = 16-c$ and $a^2+b^2 = 160-c^2$, so $2ab = (a+b)^2-(a^2+b^2) = (16-c)^2-(160-c^2) = 2c^2 - 32c + 96$, and $ab = c^2 - 16c + 48$. Then, $abc = c^3 - 16c^2 + 48c$. To find the maximum, we want the derivative of $c^3 - 16c^2 + 48c$ to be 0, so we want $3c^2 - 32c + 48 = 0$. Using the quadratic formula, we find that $c = \frac{16\pm4\sqrt{7}}{3}$. If $c = \frac{16+4\sqrt{7}}{3}$, then $abc < 0$, which is clearly not the maximum. So, we take $c = \frac{16-4\sqrt{7}}{3}$. Substituting into $abc = c^3 - 16c^2 + 48c$, we find that $abc = \frac{128}{27}(7\sqrt{7} - 10)$.

9. For a given $k > 0, n \geq 2k > 0$, consider the square R in the plane consisting of all points (x, y) with $0 \leq x, y \leq n$. Color each point in R gray if $\frac{xy}{k} \leq x + y$, and blue otherwise. Find the area of the gray region in terms of n and k.

   Solution: $2k\left(n + k\ln\frac{n-k}{k}\right)$

   First, observe that $\frac{xy}{k} \leq x+y$ iff $xy \leq kx+ky$ iff $xy-kx-ky \leq 0$ iff $(x-k)(y-k) = xy-kx-ky+k^2 \leq k^2$. So, we really want to find the area of the part of $(x-k)(y-k) \leq k^2$ that is contained in R. Notice that the graph $(x-k)(y-k) = k^2$ is just a hyperbola, and the bottom half of the hyperbola lies entirely outside the first quadrant, so we only need to look at the top half of the hyperbola.



The graph of the top half is $y = \frac{k^2}{x-k} + k$, which is strictly decreasing. Notice that $\frac{k^2}{x-k} + k = n$ for $x = \frac{k^2}{n-k} + k = \frac{nk}{n-k}$, so $\frac{k^2}{x-k} + k \geq n$ for all $x \leq \frac{nk}{n-k}$. Thus, we can express the gray area as

$$\begin{aligned}
\int_0^{\frac{nk}{n-k}} n\ dx + \int_{\frac{nk}{n-k}}^{n} \left(\frac{k^2}{x-k} + k\right)\ dx &= \frac{n^2k}{n-k} + (k^2\ln(x-k) + kx)|_{\frac{nk}{n-k}}^{n} \\
&= \frac{n^2k}{n-k} + [k^2\ln(n-k) + kn] - (k^2\ln\frac{k^2}{n-k} + k\frac{nk}{n-k}) \\
&= 2kn + k^2\ln(n-k) - k^2\ln\frac{k^2}{n-k} \\
&= 2kn + k^2\ln(n-k) - 2k^2\ln k + k^2\ln(n-k) \\
&= 2k\left(n + k\ln\frac{n-k}{k}\right)
\end{aligned}$$

10. Let $f(x) = \left(x^2 - 1\right)^n$, where n is a positive integer. Determine, in terms of n, (a, b, c), where a,b, and c are the number of distinct roots of $f^{(n)}(x)$ in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, +\infty)$, respectively.

    Solution: $(0, n, 0)$

    A polynomial of degree k has at most k roots and differentiating a polynomial decreases the multiplicity of each existing root by 1. Also, the derivative of any differentiable function with two different real 0s has a real 0 strictly between them by Rolle's theorem (the mean value theorem). This means that for $0 \leq k \leq n$, the kth derivative of f has roots of multiplicity $n-k$ at -1 and 1 and k roots of multiplicity 1 in $(-1, 1)$. So, for $k = n$, we see that there are roots of multiplicity 0 (i.e. no root) at -1 and 1, and n roots in $(-1, 1)$. Since this polynomial is of degree n, these must be the only roots.

2001 Stanford Math Tournament
Geometry

1. Find the coordinates of the points of intersection of the graphs of the equations $y = |2x| - 2$ and $y = -|2x| + 2$.

   Solution: $(1, 0), (-1, 0)$

   (x, y) is a point of intersection iff $y = |2x| - 2 = -(|2x| - 2)$, so y must be 0. Then, $|2x| = 2$, so $x = \pm 1$.

2. Jacques is building an igloo for his dog. The igloo's inside and outside are both perfectly hemispherical. The interior height at the center is 2 feet. The igloo has no door yet and contains $\frac{254}{2187}\pi$ cubic yards of hand-packed snow. What is the circumference of the igloo at its base in feet?

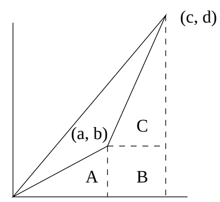
   Solution: $\frac{14}{3}\pi$

   Let R be the radius of the igloo at its base, in feet. Then, the amount of snow in the igloo is, in cubic feet, $\frac{1}{2}\left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi \cdot 2^3\right) = \frac{2}{3}\pi(R^3 - 8)$. This is equal to $\frac{254}{2187}\pi$ cubic yards, or $\frac{254}{81} = 3^3 \cdot \frac{254}{2187}$ cubic feet. Therefore, $\frac{2}{3}\pi(R^3 - 8) = \frac{254}{81}\pi$, so $R^3 = \frac{343}{27}$, and $R = \frac{7}{3}$. So, the circumference of the igloo is $2\pi R = \frac{14}{3}\pi$ feet.

3. Find the area of the convex quadrilateral whose vertices are $(0, 0), (4, 5), (9, 21), (-3, 7)$.

   Solution: $\frac{165}{2}$

   We can split this into two triangles, one with vertices $(0, 0), (4, 5), (9, 21)$ and the other with vertices $(0, 0), (9, 21), (-3, 7)$. To find the area, we will first find a formula for the area of a triangle with vertices $(0, 0), (a, b), (c, d)$ and then apply this formula to our two triangles.



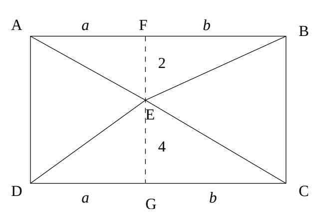
   As the figure demonstrates, we can find the area of the triangle by first finding the area of the large triangle (the one with vertices $(0, 0), (a, b), (a, 0)$) and then subtracting the area of the regions A, B, and C. The large triangle clearly has area $\frac{cd}{2}$. Region A has area $\frac{ab}{2}$, region B has area $(c - a)b$, and region C has area $\frac{(c-a)(d-b)}{2}$. So, the area of the triangle is $\frac{cd}{2} - \frac{ab}{2} - (c - a)b - \frac{(c-a)(d-b)}{2} = \frac{ad-bc}{2}$. Of course, the area should only be positive, so we need to take the absolute value of this. (If you're familiar with determinants, think about what this means in terms of determinants.)

   Using the formula, the area of the triangle with vertices $(0, 0), (4, 5), (9, 21)$ is $\frac{|4 \cdot 21 - 5 \cdot 9|}{2} = \frac{39}{2}$ and the area of the triangle with vertices $(0, 0), (9, 21), (-3, 7)$ is $\frac{|9 \cdot 7 + 21 \cdot 3|}{2} = 63$. So, the total area of the quadrilateral is $\frac{165}{2}$.

4. E is a point in the interior of rectangle $ABCD$. $AB = 6$, triangle ABE has area 6, and triangle CDE has area 12. Find $(EA)^2 - (EB)^2 + (EC)^2 - (ED)^2$.

   Solution: 0

   Let l be the line through E perpendicular to AB and CD. Let F and G be the intersections of l with AB and CD, respectively. Then, $AF = DG$, call this a, and $BF = CG$, call this b. Since triangle ABE has area 6, $EF = 2$. Since triangle CDE has area 12, $EG = 4$.



By the Pythagorean Theorem, we have the following equalities:

$$a^2 + 4 = (EA)^2, b^2 + 4 = (EB)^2, b^2 + 16 = (EC)^2, a^2 + 16 = (ED)^2$$

Thus, $(EA)^2 - (EB)^2 + (EC)^2 - (ED)^2 = 0$.

5. Two identical cones, each 2 inches in height, are held one directly above another with the pointed end facing down. The upper cone is completely filled with water. A small hole is punctured in the bottom of the upper cone so that the water trickles down into the bottom cone. When the water reaches a depth of 1 inch in the bottom cone, what is its depth in the upper cone?

   Solution: $\sqrt[3]{7}$

   Let r be the radius of the base of one cone. Then, the amount of water is $V = \frac{1}{3}2\pi r^2 = \frac{2}{3}\pi r^2$. When the height of the water in the bottom cone is 1, using similar triangles, the volume of water in the bottom cone is $V_b = \frac{1}{3}1\pi\left(\frac{r}{2}\right)^2 = \frac{1}{12}\pi r^2$. So, the volume of water is the upper cone is $V_u = V - V_b = \frac{7}{12}\pi r^2 = \frac{1}{3}h_{new}\pi r_{new}^2$ where h_{new} is the depth of the water in the upper cone and r_{new} is the radius of the cone formed by the remaining water. Again, by similar triangles, $\frac{r_{new}}{r} = \frac{h_{new}}{2}$. Substituting,

$$V_u = \frac{7}{12}\pi r^2 = \frac{1}{3}\pi h_{new}\left(r\frac{h_{new}}{2}\right)^2.$$

   Solving this equation, we find that $h_{new} = \sqrt[3]{7}$.

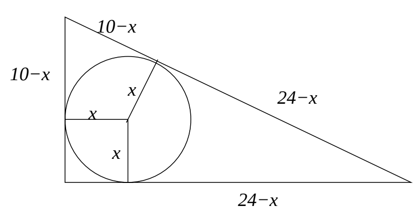
6. Find the radius of a circle inscribed in the triangle determined by the lines $4x + 3y = 24$, $56x - 33y = -264$, and $3x - 4y = 18$.

   Solution: 4

   First, observe that the lines $4x + 3y = 24$ and $3x - 4y = 18$ are perpendicular, since their slopes are negative reciprocals. We can also find their intersection point; $4x + 3y = 24$ can be rewritten as $y = \frac{24-4x}{3}$, while $3x - 4y = 18$ can be rewritten as $y = \frac{3x-18}{4}$. Setting these two equal, we have $\frac{24-4x}{3} = \frac{3x-18}{4}$. Cross-multiplying, $4(24 - 4x) = 3(3x - 18)$, so $96 - 16x = 9x - 54$, and $150 = 25x$, and $x = 6$. Then, $y = \frac{24-4x}{3} = 0$. Thus, one vertex of the triangle is $(6, 0)$. Similarly, we find that the other two vertices of the triangle are $(0, 8)$ and $(-\frac{66}{5}, -\frac{72}{5})$.

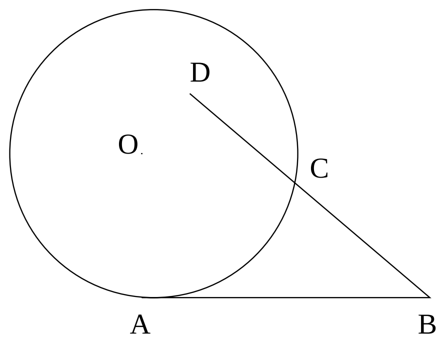
   Now, we must find the lengths of the legs of the triangle. The leg between the vertices $(6, 0)$ and $(0, 8)$ has length $\sqrt{(6-0)^2 + (0-8)^2} = 10$, while the other leg has length 24. Thus, the hypotenuse of the triangle has length $\sqrt{10^2 + 24^2} = 26$.

   Let the radius of the incircle be x. Using the fact that the tangents from a point to a circle have the same length, we have the following diagram:



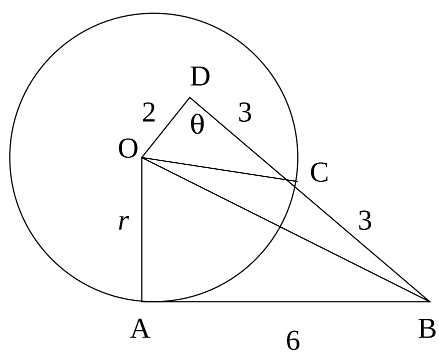
   Thus, $(10 - x) + (24 - x) = 26$, so $x = 4$.

7. In the figure, AB is tangent at A to the circle with center O; point D is interior to the circle; and DB intersects the circle at C. If $BC = DC = 3$, $OD = 2$ and $AB = 6$, then find the radius of the circle.



Solution: $\sqrt{22}$

Let $\theta = \angle ODB$ and r be the radius of circle O.



Observe that ABO is a right triangle, so by the Pythagorean theorem. $BO^2 = 36 + r^2$. By the law of cosines with triangle ODC, $r^2 = 4 + 9 - 12\cos\theta$, so $-12\cos\theta = r^2 - 13$. By the law of cosines with triangle ODB, $BO^2 = 4 + 36 - 24\cos\theta = 40 + 2(r^2 - 13) = 14 + 2r^2$. Therefore, $36 + r^2 = 14 + 2r^2$, so $r^2 = 22$, and $r = \sqrt{22}$.

8. Let S be the solid tetrahedron with boundary points $(0,0,0), (2,4,0), (5,1,0), (3,2,10)$. Let $z_1 = \max\{q \mid (\frac{12}{5}, \frac{23}{10}, q) \in S\}$ and let $z_2 = \max\{r \mid (\frac{9}{2}, \frac{5}{4}, r) \in S\}$. Find $z_1 - z_2$.

   Solution: $\frac{15}{4}$

   A tetrahedron is made up of four planes. Notice that z_1 and z_2 must lie on the boundary of S, so each lies on one of the four planes. However, we can ignore the plane defined by $(0,0,0), (2,4,0), (5,1,0)$ because z_1 and z_2 will not be on this plane (as it is lower than all the rest, and we want the maximum). Also notice that a vertical line intersects each of the other 3 planes exactly once, but only the lowest intersection point is actually on the tetrahedron.

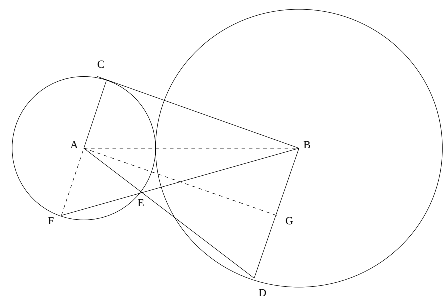
   To find the planes, recall that any plane that doesn't go through the origin can be represented as $ax + by + cz = 1$. Let P_1 be the plane defined by $(2,4,0), (5,1,0), (3,2,10)$. Then, $2a + 4b = 1$, $5a + b = 1$, and $3a + 2b + 10c = 1$. Solving, we find that P_1 is the plane $\frac{1}{6}x + \frac{1}{6}y + \frac{1}{60}z = 1$. A plane that does go through the origin can be written as $x + by + cz = 0$. Let P_2 be the plane defined by $(0,0,0), (2,4,0), (3,2,10)$. Then, using the same method as above, we find that the plane is $x - \frac{1}{2}y - \frac{1}{5}z = 0$. Let P_3 be the plane defined by $(0,0,0), (5,1,0), (3,2,10)$. Then, it has equation $x - 5y + \frac{7}{10}z = 0$.

   Using these equations, we find that $(\frac{12}{5}, \frac{23}{10}, 13)$ is on P_1, $(\frac{12}{5}, \frac{23}{10}, \frac{25}{4})$ is on P_2, and $(\frac{12}{5}, \frac{23}{10}, 13)$ is on P_3. Only the lowest is on the tetrahedron, so $z_1 = \frac{25}{4}$. Similarly, we find that $z_2 = \frac{5}{2}$, so $z_1 - z_2 = \frac{15}{4}$.

9. Circles A and B are tangent and have radii 1 and 2, respectively. A tangent to circle A from the point B intersects circle A at C. D is chosen on circle B so that AC is parallel to BD and the two segments BC and AD do not intersect. Segment AD intersects circle A at E. The line through B and E intersects circle A through another point F. Find EF.

Solution: $2\sqrt{3}/3$

Draw a line from A perpendicular to BD. Let G the point of intersection of this line with BD. Now, notice that $ACBG$ is a rectangle.



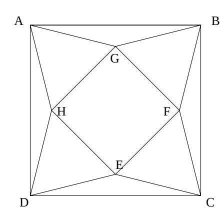
Since C is on circle A, $AC = 1$. Also, $AB = 3$ since circle A has radius 1 and circle B has radius 2. By the pythagorean theorem, $BC = AG = 2\sqrt{2}$. Then, AGD is a right triangle with legs 1 and $2\sqrt{2}$, so $AD = 3$. Then, since A, E, and D are collinear, $AD = AE + ED$, so $ED = 2$. This means that $ED = BD = 2$, so DEB is an isosceles triangle with vertex D. Notice that AEF is also isosceles, since E and F are both on circle A. Also, $\angle AEF = \angle BED$, so the triangles AEF and DEB are similar. In particular, the ratio of a side of DEB to a side of AEF is 2 to 1 because $BD = 2$ and $AF = 1$, so $FE = \frac{1}{2}EB$, and $FE = \frac{1}{3}BF$.

So, now we just need to find BF. Let $\theta = \angle AFE$. Then, $\angle AEF = \theta$, so $\angle FAE = 180° - 2\theta$. Also, since AEF and BED are similar, $\angle BDE = 180° - 2\theta$. Since AGD is a right triangle, $\angle GAD = 2\theta - 90°$, so $\angle GAF = \angle GAD + \angle FAE = (2\theta - 90°) + (180° - 2\theta) = 90°$. Since CAG is also a right angle, this means that C, A, and F are collinear. So, BCF is a right triangle, and it has legs 2 and $2\sqrt{2}$. So, $BF = 2\sqrt{3}$, and $FE = \frac{1}{3}BF = \frac{2\sqrt{3}}{3}$.

10. E is a point inside square $ABCD$ such that $\angle ECD = \angle EDC = 15°$. Find $\angle AEB$.

Solution: 60°

We will use symmetry to solve this problem. Place points F, G, and H inside the square so that triangles EDC, FCB, GBA, and HAD are congruent.



Then triangles EDH, FCE, GBF, and GAH are isosceles with vertex angle 60°, so they are equilateral. So, $GHEF$ is a rhombus. Then, since $\angle DEC$ is 150° and $\angle DEH$ and $\angle CEF$ are both 60°, $\angle HEF$ is 90°. By symmetry, the other angles of $GHEF$ are also 90°, so $GHEF$ is a square.

Since AHG is equilateral and $GHEF$ is a square, AEH is an isosceles triangle with vertex angle 150°. Thus, $\angle AEH$ is 15°. Similarly, $\angle BEF$ is 15°, so $\angle AEB$ is 60°.

2001 Stanford Math Tournament
General Test

1. If $a \star b$ is defined as $2a - b^a$, what value is associated with $3 \star 2$?

   Solution: $\mathbf{-2}$

   $3 \star 2 = 2 \cdot 3 - 2^3 = 6 - 8 = -2$.

2. Find the coordinates of the points of intersection of the graphs of the equations $y = |2x| - 2$ and $y = -|2x| + 2$.

   Solution: $\mathbf{(1, 0), (-1, 0)}$

   (x, y) is a point of intersection iff $y = |2x| - 2 = -(|2x| - 2)$, so y must be 0. Then, $|2x| = 2$, so $x = \pm 1$.

3. ABCD is a 4-digit number. What is the largest number that can be formed with AB a prime 2-digit number and C and D each a prime 1-digit number?

   Solution: 9777

   The largest 2-digit prime number is 97, and the largest 1-digit prime is 7, so the largest number with the specified properties is 9777.

4. Let A be the set of all non-composite positive integers. Let B be the set of all squares of integers. Let C be the set of all multiples of 3. What is the size of $A \cap (B \cup C)$?

   Solution: 2

   $B \cup C =$ all squares and multiples of 3, so $A \cap (B \cup C) = \{1, 3\}$, and the size of this is **2**.

5. What is i^{2959}?

   Solution: $\boldsymbol{-i}$

   Notice that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Since $2959 = 4 \times 739 + 3$, $i^{2959} = (i^4)^{739} i^3 = i^3 = -i$.

6. The points $Q = (9, 14)$ and $R = (a, b)$ are symmetric with respect to the point $P = (5, 3)$. What are the coordinates of point R?

   Solution: $\mathbf{(1, -8)}$

   P is the midpoint of the segment QR, so $\frac{a+9}{2} = 5$ and $\frac{b+14}{2} = 3$. Thus, $a = 1$ and $b = -8$, so $R = (1, -8)$.

7. If $F(x) = 3x^3 - 2x^2 + x - 3$, find $F(1 + i)$.

   Solution: $\boldsymbol{-8 + 3i}$

   Notice that $(1 + i)^2 = 2i$, so $(1 + i)^3 = 2i - 2$. So, $3(1 + i)^3 - 2(1 + i)^2 + (1 + i) - 3 = 3(2i - 2) - 2(2i) + (1 + i) - 3 = -8 + 3i$.

8. Express the absolute value of the difference between $0.\overline{36}$ and 0.36 as a common fraction.

   Solution: 1/275

   Let $x = 0.\overline{36}$. Then, $100x = 36.\overline{36}$, so $99x = 36$, and $x = \frac{36}{99} = \frac{4}{11}$. So, the difference between $0.\overline{36}$ and 0.36 is $\frac{4}{11} - \frac{9}{25} = \frac{1}{275}$.

9. Jonathan chooses 10 cards without replacement from a standard 52-card deck of cards (without jokers). What is the probability that he does <u>not</u> draw the 3 of clubs <u>and</u> he does <u>not</u> draw the King of spades?

   Solution: $\mathbf{\frac{287}{442}}$

   The number of ways to draw 10 cards that are not either one of those cards is $\binom{50}{10}$, and the total number of ways to draw 10 cards is $\binom{52}{10}$. So, the probability is $\frac{\binom{50}{10}}{\binom{52}{10}} = \frac{\frac{50!}{10!40!}}{\frac{52!}{10!42!}} = \frac{42 \cdot 41}{52 \cdot 51} = \mathbf{\frac{287}{442}}$.

10. Julie has a 12 foot by 20 foot garden. She wants to put fencing around it to keep out the neighbor's dog. Normal fenceposts cost \$2 each while strong ones cost \$3 each. If Julie needs one fencepost for every 2 feet and has \$70 to spend on fenceposts, what is the largest number of strong fenceposts she can buy?

    Solution: 6

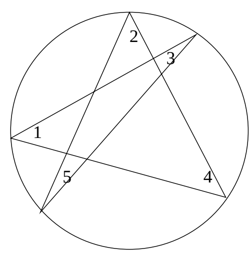
    Let x be the number of normal fenceposts and y be the number of strong fenceposts. The perimeter of Julie's garden is 64 feet, and she needs one fencepost every 2 feet, so $x + y = 32$. The total cost of the fenceposts is $2x + 3y$, and we want $2x + 3y \leq 70$. Since $2x + 2y = 64$, we find that $y \leq 6$.

11. Anne has a cube that she wants to paint. If she decides to paint each face a different color and she has 6 colors, how many distinct ways can she paint the cube? (If one painted cube can be rotated to get another, they are the same.)

    Solution: 30

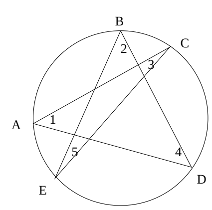
    Let's label the colors 1 through 6. Anne wants to paint each face a different color, so she will use each color exactly once. Since rotated paintings are the same, let's always rotate our painting so that color 1 is on the bottom. Then, there are 5 colors, and 5 faces left. She can color these in $5! = 120$ ways. However, notice that she can spin the cube around (always keeping color 1 on the bottom), and these painting are all the same. There are 4 such rotations, so there are actually only $\frac{120}{4} = 30$ ways to paint the cube.

12. Find, in degrees, the sum of angles 1, 2, 3, 4, and 5 in the star-shaped figure shown.



    Solution: 180°

    Let's label the points where the star meets the circle A, B, C, D, and E.



    The intercepted arc of angle 1 is CD, so the measure of angle 1 is half the arc-length of CD. We can do the same for the other angles, so the sum of the angles is half the sum of the arc lengths AB, BC, CD, DE, and EA. These make up the whole circle, so their sum is 360°. Therefore, the sum of angles 1, 2, 3, 4, and 5 is 180°.

13. Each valve A, B, and C, when open, releases water into a tank at its own constant rate. With all three valves open, the tank fills in 1 hour, with only valves A and C open it takes 1.5 hours, and with only valves B and C open it takes 2 hours. How many hours will it take to fill the tank with only valves A and B open?

    Solution: $\mathbf{1.2 = \frac{6}{5}}$

    Let r_A, r_B, r_C be the rates of valves A, B, and C in tanks per hour, respectively. Then, $1(r_A+r_B+r_C) = 1$, $1.5(r_A + r_C) = 1$, and $2(r_B + r_C) = 1$. So, $r_A + r_B + r_C = 1$, $r_A + r_C = \frac{2}{3}$, and $r_B + r_C = \frac{1}{2}$. So, $r_A + r_B = 2(r_A + r_B + r_C) - (r_A + r_C) - (r_B + r_C) = 2 - \frac{2}{3} - \frac{1}{2} = \frac{5}{6}$. Thus, it takes $\frac{6}{5} = 1.2$ hours to fill the tank with only valves A and B open.

14. Lattice paths are paths consisting of one unit steps in the positive horizontal or vertical directions. How many distinct lattice paths are there from the origin to the point $(5, 4)$?

    Solution: 126

    To get to $(5, 4)$ from the origin, you must go to the right 5 times and up 4 times. The four vertical steps can be taken in any order in the 9 total steps, and thus there are $\binom{9}{4} = 126$ distinct lattice paths.

15. Three circles, each of area 4π, are all externally tangent. Their centers form a triangle. What is the area of the triangle?

    Solution: $4\sqrt{3}$

    Each circle has radius 2, so the triangle is equilateral with side length 4. Thus, the area of the triangle is $4\sqrt{3}$.

16. Two equilateral triangles sharing an edge have a combined area of π. What is the square of the length of their shared edge?

    Solution: $\frac{2\pi\sqrt{3}}{3}$

    Let s be the length the shared edge. An equilateral triangle of side length s has area $\frac{s^2\sqrt{3}}{4}$, so $2\frac{s^2\sqrt{3}}{4} = \pi$, and $s^2 = \frac{2\pi\sqrt{3}}{3}$.

17. Frogger wants to cross a stream. He starts on one bank and jumps onto the first passing log, which is traveling at 9 feet per second. The following logs are at speeds such that the second is twice is fast as the first, the third is twice as fast as the second, the fourth is one-third as fast as the third, and the last (fifth) is one-third as fast as the fourth (all of the logs travel in the same direction). If there is a 2 second interval between jumps, then how far does he travel down river from his original point, once he reaches the opposite bank?

    Solution: 158

    The first log travels at 9 feet per second, so the second travels at 18 feet per second, the third travels at 36 feet per second, the fourth travels at 12 feet per second, and the last travels at 4 feet per second. Frogger spends 2 seconds on each log, so he travels $2(9 + 18 + 36 + 12 + 4) = 158$ feet.

18. Rice tuition was \$7200 one semester. Each student takes a minimum of 12 credits and a maximum of 20 credits. For each credit, a class meets for 3 hours a week. A semester is 15 weeks with no breaks. What is the difference in cost in dollars per hour of class that a student taking the maximum load and a student taking the minimum load pays, to the nearest penny?

    Solution: \$5.33

    A student taking the minimum load has $12 \cdot 3 \cdot 15 = 540$ class hours over the semester so pays $\frac{7200}{540} = \frac{40}{3} = \$13.33$ per class hour. A student taking the maximum load has $20 \cdot 3 \cdot 15 = 900$ class hours so pays $\frac{7200}{900} = \$8$ per class hour. So, the difference is $\$13.33 - \$8 = \$$**5.33**.

19. Suppose $a_1, a_2, a_3, \ldots$ is a sequence of numbers such that $a_1 = 1$ and $a_{n+1} = a_n + (2n + 1)$ for all positive integers n. Find a_{20}.

    Solution: 400

    In general, $a_n = n^2$. This is certainly true for a_1, and if $a_n = n^2$, then $a_{n+1} = a_n + (2n + 1) = n^2 + 2n + 1 = (n + 1)^2$, so it is true for all n (this sort of proof is called induction).

20. My spouse and I have 9 kids. Each child gets married and has exactly 9 children of their own. By the time I have 100,000 descendants (including every generation) what is the longest title that applies to me? (Assuming no married blood relatives.)

    a. grandparent

    b. great grandparent

    c. great great grandparent

d. great great great grandparent

e. great great great great grandparent

f. great great great great great grandparent

g. great great great great great great grandparent

Solution: great great great great grandparent

I have 9 children, 9^2 grandchildren, 9^3 great grandchildren, etc. So, the number of descendants I have in the n-th generation is $9 + 9^2 + \ldots + 9^n = \frac{9^{n+1}-1}{8} - 1$. The smallest value of n for which this is at least 100,000 is 6, which makes me a great great great great grandparent.

21. Find the greatest integer x for which $3^{20} > 32^x$.

Solution: 6

First, notice that $32^x = 2^{5x}$, so $3^{20} > 32^x$ if and only if $3^4 > 2^x$. $3^4 = 81$, and it is easy to check that 2^6 is the largest power of 2 less than 81.

22. If $x + y = xy$, with x and y real, what value can x not have?

Solution: 1

$x + y = xy$ if and only if $xy - x - y + 1 = 1$, or $(x-1)(y-1) = 1$. This is impossible if $x = 1$.

23. Instead of using two standard cubical dice in a board game, three standard cubical dice are used so that the game goes more quickly. In the regular game, doubles are needed to get out of the "pit". In the revised game, doubles or triples will get you out. How many times as likely is it for a player to get out of the "pit" on one toss under the new rules as compared to the old rules?

Solution: 8/3

The probability of rolling two different numbers with two different dice is $\frac{6 \cdot 5}{6 \cdot 6} = \frac{5}{6}$, so the probability of rolling doubles is $1 - \frac{5}{6} = \frac{1}{6}$. The probability of rolling three different numbers with three different dice is $\frac{6 \cdot 5 \cdot 4}{6 \cdot 6 \cdot 6} = \frac{5}{9}$, so the probability of rolling doubles or triples with three dice is $1 - \frac{5}{9} = \frac{4}{9}$. So, the ratio of the two is $\frac{4}{9} / \frac{1}{6} = \frac{8}{3}$.

24. A circle is drawn with center at the origin and radius 2.5. Find the coordinates of all intersections of the circle with an origin-centered square of side length 4 whose sides are parallel to the coordinate axes.

Solution: $(\pm 1.5, \pm 2), (\pm 2, \pm 1.5)$

The square consists of all points (x, y). with $-2 \leq x, y \leq 2$ and at least one of x and y equal to 2 or -2. The points on the circle are all points (x, y) with $x^2 + y^2 = 6.25$. So, the intersection of these two is $(\pm 2, \pm 1.5)$ and $(\pm 1.5, \pm 2)$.

25. Write $20_{10} \frac{(70_{17})!}{(170_8)!}$ as a decimal in base 6. The subscript indicates the base in which the number is written (i.e., 15_6 is 15 base 6, so $15_6 = 11_{10}$)

Solution: 0.1_6 or 0.1

The easiest way to solve this problem is to convert everything into base 10.

$$\begin{aligned} 70_{17} &= 7 \times 17^1 + 0 \times 17^0 = 119 \\ 170_8 &= 1 \times 8^2 + 7 \times 8^1 + 0 \times 8^0 = 120 \\ 20_{10} \frac{(70_{17})!}{(170_8)!} &= 20 \frac{119!}{120!} = \frac{1}{6} \end{aligned}$$

Converting to base 6, $\frac{1}{6} = 0.1_6$.

26. Jacques is building an igloo for his dog. The igloo's inside and outside are both perfectly hemispherical. The interior height at the center is 2 feet. The igloo has no door yet and contains $\frac{254}{2187}\pi$ cubic yards of hand-packed snow. What is the circumference of the igloo at its base in feet?

Solution: $\boldsymbol{\frac{14}{3}\pi}$

Let R be the radius of the igloo at its base, in feet. Then, the amount of snow in the igloo is, in cubic feet, $\frac{1}{2}\left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi \cdot 2^3\right) = \frac{2}{3}\pi(R^3-8)$. This is equal to $\frac{254}{2187}\pi$ cubic yards, or $\frac{254}{81} = 3^3 \cdot \frac{254}{2187}$ cubic feet. Therefore, $\frac{2}{3}\pi(R^3-8) = \frac{254}{81}\pi$, so $R^3 = \frac{343}{27}$, and $R = \frac{7}{3}$. So, the circumference of the igloo is $2\pi R = \frac{14}{3}\pi$ feet.

27. How many solutions are there to $x^7 + y^7 = z^7$ with x, y, z real?

Solution: $\boldsymbol{\infty}$

For any real x, y, $x^7 + y^7$ is real, so $x^7 + y^7$ has a 7-th root. If we let z be this root, then (x, y, z) is a solution to $x^7 + y^7 = z^7$. So, for any x, y, there is a solution. There are infinitely many x, y, so there must be infinitely many solutions.

28. Find all prime factors of $3^{18} - 2^{18}$.

Solution: $\mathbf{5, 7, 19, 577, 1009}$

$$\begin{aligned} 3^{18} - 2^{18} &= (3^9 - 2^9)(3^9 + 2^9) = (3^3 - 2^3)(3^6 + 3^3 \cdot 2^3 + 2^6)(3^3 + 2^3)(3^6 - 3^3 \cdot 2^3 + 2^6) \\ &= 19 \cdot 1009 \cdot 35 \cdot 577 = 5 \cdot 7 \cdot 19 \cdot 577 \cdot 1009 \end{aligned}$$

29. If $a \neq 1$ and $\sqrt[a]{10000_a} = 10_a$, find a. The subscript indicates the base in which the number is written.

Solution: 4

$10000_a = a^4$ and $10_a = a$, so $\sqrt[a]{a^4} = a$. Thus, $a^{4/a} = a$, and $4/a = 1$, so $a = 4$.

30. Suppose 100 students have at least one of math, applied math, or statistics as one of their majors. There are 24 statistics majors, 46 applied math majors, and 55 pure math majors. There are 14 who are at least pure & applied math, 10 at least applied math & statistics and 7 at least statistics & pure math. How many triple majors (people with all three majors) are there, if any?

Solution: 6

Using the principle of inclusion-exclusion, the number of triple majors is given by $100 - (46 + 24 + 55) + (14 + 10 + 7) = 6$.

31. The sum of 3 real numbers is 0. If the sum of their cubes is π^e, what is their product?

Solution: $\boldsymbol{\frac{\pi^e}{3}}$

Let x, y, and z be the 3 real numbers. Then $x + y + z = 0$, so $z = -(x + y)$. Also,

$$\begin{aligned} \pi^e &= x^3 + y^3 + z^3 = x^3 + y^3 - (x+y)^3 \\ &= -3x^2y - 3xy^2 = -3xy(x+y) = 3xyz \end{aligned}$$

So, $xyz = \frac{1}{3}\pi^e$.

32. Given a real-valued function $f(x) = \sqrt{\frac{x+3}{|2x+6|}}$, what is the least integer value which lies in the domain of the function?

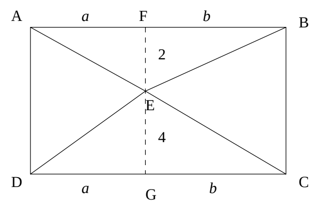
Solution: $\mathbf{-2}$

We have $x \geq -3$, as the value under the radical must be nonnegative, and the absolute value is non negative, so $x + 3 \geq 0$. But, the denominator must be non-zero, so we must exclude $x = -3$, as then $2x + 6 = 0$. Hence, the least integer value is -2.

33. E is a point in the interior of rectangle $ABCD$. $AB = 6$, triangle ABE has area 6, and triangle CDE has area 12. Find $(EA)^2 - (EB)^2 + (EC)^2 - (ED)^2$.

Solution: 0

Let l be the line through E perpendicular to AB and CD. Let F and G be the intersections of l with AB and CD, respectively. Then, $AF = DG$, call this a, and $BF = CG$, call this b. Since triangle ABE has area 6, $EF = 2$. Since triangle CDE has area 12, $EG = 4$.



By the Pythagorean Theorem, we have the following equalities:

$$a^2 + 4 = (EA)^2, b^2 + 4 = (EB)^2, b^2 + 16 = (EC)^2, a^2 + 16 = (ED)^2$$

Thus, $(EA)^2 - (EB)^2 + (EC)^2 - (ED)^2 = 0$.

34. What is the radius of the incircle of a right triangle with legs of lengths 7 and 24?

Solution: 3

Let ABC be a right triangle with $AB = 24$ and $BC = 7$. Then, $AC = 25$. Let x be the radius of the incircle of ABC. Let D, E, and F be the points on AB, BC, and AC, respectively, that are tangent to the incircle. Then, $AD = AF = 24 - x$ and $CE = CF = 7 - x$. So, $AC = 31 - 2x = 25$, and $x = 3$.

35. If $\log_A B + \log_B A = 3$ and $A < B$, find $\log_B A$.

Solution: $\frac{3-\sqrt{5}}{2}$

Let $x = \log_B A$. Then $B^x = A$, so $B = A^{\frac{1}{x}}$. Thus, $\log_A B = \frac{1}{x}$. So, $x + \frac{1}{x} = 3$. Multiplying through by x, $x^2 + 1 = 3x$, so $x^2 - 3x + 1 = 0$. By the quadratic formula,

$$x = \frac{3 \pm \sqrt{5}}{2}$$

Since $A < B$, $\log_B A < 1$, so

$$x = \frac{3 - \sqrt{5}}{2}$$

.

36. ABC is an equilateral triangle with edge of length 256. Let n be the maximum number of non-overlapping equilateral triangles of edge length $\frac{1}{2}$ that can be fit into ABC. What is the prime factorization of n?

Solution: 2^{18}

For a triangle of side length n, we can fit in 4 triangles with sides of length $\frac{n}{2}$ (maybe draw a figure). $\frac{256}{2^9} = \frac{1}{2}$, so we can do this 9 times. Each time, the number of triangle increases 4-fold. $4^9 = 2^{18}$.

37. If $x^2 + \frac{1}{x^2} = 7$, find $\left|x^3 + \frac{1}{x^3}\right|$.

Solution: 18

Observe that $(x + \frac{1}{x})^2 = x^2 + 2 + \frac{1}{x^2} = 9$. so $\left|x + \frac{1}{x}\right| = 3$. Cubing, $\left|x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right| = 27$. Now, $x^3 + \frac{1}{x^3}$ has the same sign as $3x + \frac{3}{x}$ because if x is positive, both are positive, and if x is negative, both are negative. So, $\left|x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right| = \left|x^3 + \frac{1}{x^3}\right| + 3\left|x + \frac{1}{x}\right| = 27$. Since $\left|x + \frac{1}{x}\right| = 3$, $\left|x^3 + \frac{1}{x^3}\right| = 18$.

38. Eleven pirates find a treasure chest. When they split up the coins in it evenly among all the pirates, they find that there are 5 coins left. They throw one pirate overboard and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split perfectly. What is the least number of coins there could have been?

Solution: 423

Let n be the number of coins in the chest. Then, $n \equiv 5 \pmod{11}$ (this means that the remainder of n when divided by 11 is 5), $n \equiv 3 \pmod{10}$, and $n \equiv 0 \pmod{9}$. The Chinese Remainder Theorem gives an elegant solution to this problem, or you can solve it as follows. Since $n \equiv 3 \pmod{10}$, we know that the last digit of n must be 3. Also, since $n \equiv 0 \pmod{9}$, n is divisible by 9, so $n = 9q$ for some q. Since n ends in a 3, q must end in a 7. That means that q looks like $10p + 7$, so $n = 9(10p + 7) = 90p + 63$. Now, you can just try $p = 1, 2, \ldots$, and you'll find that $p = 4$ works.

2001 Stanford Math Tournament
Team Test

1. $ABCD$ is a square with sides of unit length. Points E and F are taken on sides AB and AD respectively so that $AE = AF$ and the quadrilateral $CDFE$ has maximum area. What is this maximum area?

   Solution: 5/8

   Let $x = AE$. Then, $BE = 1 - x$, so the area of $CDFE$ is $1 - \frac{x^2}{2} - \frac{1-x}{2} = 1 - \frac{x^2-x+1}{2}$, which is maximized when $x^2 - x + 1$ is minimized. $x^2 - x + 1$ is minimized at $x = \frac{1}{2}$. Plugging this into our formula for the area of $CDFE$, we find that the maximum area of $CDFE$ is $\frac{5}{8}$.

2. How many positive integers between 1 and 400 (inclusive) have exactly 15 positive integer factors?

   Solution: 3

   In general, if a number has prime factorization $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, then its positive integer factors are exactly those of the form $p_1^{j_1} p_2^{j_2} \ldots p_k^{j_k}$ where $0 \leq j_i \leq n_i$ for each i between 1 and k, inclusive. In particular, there are $n_i + 1$ choices for j_i, so there are $(n_1+1)(n_2+1)\cdots(n_k+1)$ positive integer factors. In our case, since $15 = 1 \cdot 15 = 3 \cdot 5$ are the only factorizations, either $n_1 = 0$ and $n_2 = 14$ or $n_1 = 2$ and $n_2 = 4$. So, a positive integer has exactly 15 positive integers if and only if it is of the form p^{14} for some prime p or of the form $p_1^2 p_2^4$ for two distinct primes p_1 and p_2. $2^{14} > 400$, so there are no numbers of the first form between 1 and 400. The only ones of the second form that are between 1 and 400 are $3^2 \cdot 2^4 = 144$, $5^2 \cdot 2^4 = 400$, and $2^2 \cdot 3^4 = 324$. Therefore, there are 3 positive integers between 1 and 400, inclusive, that have exactly 15 positive integer factors.

3. Find the 2000th positive integer that is not the difference between any two integer squares.

   Solution: 7998

   First, we will try to find all numbers that *can* be expressed as a difference between two integer squares. For any integer n, $(n+1)^2 - n^2 = 2n+1$, so any odd number can be expressed as the difference between two integer squares. Similarly, $(n+2)^2 - n^2 = 4n+4 = 4(n+1)$, so any number divisible by 4 can be expressed as the difference between two integer squares.

   What about the other numbers, the ones of the form $4n+2$? It is easy to see that none of these can be the difference between two integer squares. Assume that it could; then $x^2 - y^2 = 4n+2$ for some x and y. Then, either x and y are both even, or they are both odd, since their difference is even. However, then $x^2 - y^2$ would be divisible by 4, while $4n+2$ is not. So, the integers that cannot be expressed as the difference between two integer squares are exactly those of the form $4n+2$. The first is $4(0)+2 = 2$, so the 2000^{th} one is $4(1999)+2 = 7998$.

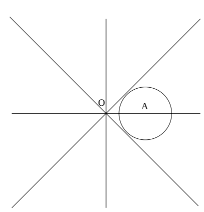
4. For what values of a does the system of equations

   $$x^2 = y^2, (x-a)^2 + y^2 = 1$$

   have exactly 2 solutions?

   Solution: $\pm\sqrt{2}$

   We can plot this system to get a rough idea of what it looks like. $x^2 = y^2$ is the same as $x = \pm y$, or a pair of perpendicular lines. $(x-a)^2 + y^2 = 1$ is a circle with center $(a, 0)$ and radius 1. We can see that the system will have exactly 2 solutions if and only if the circle is tangent to both lines:



Let's label the origin O, the center of the circle $A = (a, 0)$, and the two points of tangency B and C. Then, $OBAC$ is a rectangle, and $AB = AC = 1$. Thus, $OBAC$ must be a square, and AO, its diagonal, must have length $\sqrt{2}$. Thus, $a = \sqrt{2}$. Now, of course, the picture is exactly the same if the circle is on the other side of the origin, so $a = -\sqrt{2}$ is also a possible solution. Thus, the values of a giving exactly 2 solutions are $\pm\sqrt{2}$.

5. What quadratic polynomial whose coefficient of x^2 is 1 has roots which are the complex conjugates of the solutions of $x^2 - 6x + 11 = 2xi - 10i$? (Note that the complex conjugate of $a + bi$ is $a - bi$, where a and b are real numbers.)

   Solution: $\boldsymbol{x^2 + (-6 + 2i)x + (11 - 10i)}$

   Let a, b be the solutions of $x^2 - 6x + 11 = 2xi - 10i$. Then, a and b are the roots of $x^2 - (6+2i)x + (11+10i)$, so $x^2 - (6 + 2i)x + (11 + 10i) = (x - a)(x - b) = x^2 - (a + b)x + ab$. Therefore, $a + b = 6 + 2i$ and $ab = 11 + 10i$.

   Let $\overline{a}, \overline{b}$ the complex conjugates of a, b, respectively. Then, the quadratic polynomial with one as the coefficient of x^2 and $\overline{a}, \overline{b}$ as roots is $(x - \overline{a})(x - \overline{b}) = x^2 - (\overline{a} + \overline{b})x + \overline{a}\overline{b} = x^2 - (\overline{a + b})x + \overline{ab} = x^2 - (6 - 2i)x + (11 - 10i) = x^2 + (-6 + 2i)x + (11 - 10i) = 0$.

6. Find the least n such that any subset of $\{1, 2, \ldots, 100\}$ with n elements has 2 elements with a difference of 9.

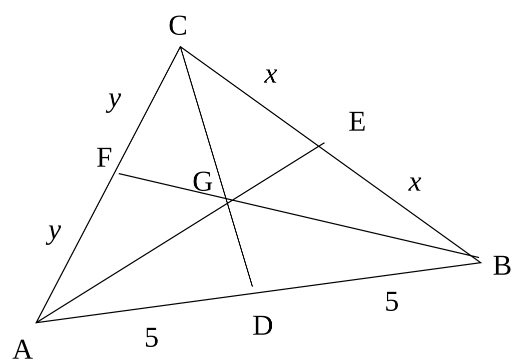
   Solution: 55

   Let $A = \{18a + 1, 18a + 2, \ldots, 18a + 9 : a = 0, 1, 2, 3, 4, 5\}$. Then, notice that A has 54 elements, and no 2 elements have a difference of 9. So, we know $n \geq 55$. Now, we just need to convince ourselves that any subset with 55 elements must have 2 elements with a difference of 9. To do this, take the sets $\{x, x + 9\}$ for $x = 1, 2, \ldots, 91$. Notice that there are 91 such sets. The numbers $1, 2, \ldots, 9$ and $92, \ldots, 100$ each fall into exactly one of the set, while all other numbers fall into two of them. So, there are 18 numbers that fall into exactly one set, and then there are 73 other sets. That means that, in a set of 55 elements, at least 37 fall into two sets. As $37 \cdot 2 + 18 = 92 > 91$ (the number of sets), by the pigeonhole principle, two of them fall in the same set, and thus have a difference of 9.

7. The median to a 10 cm side of a triangle has length 9 cm and is perpendicular to a second median of the triangle. Find the exact value in centimeters of the length of the third median.

   Solution: $\boldsymbol{3\sqrt{13}}$

   Call the triangle ABC, where $AB = 10$, $BC = 2x$, and $CA = 2y$. Let D, E, and F be the midpoints of AB, BC, and CA, respectively. Let G be the centroid (the point where the three medians intersect).



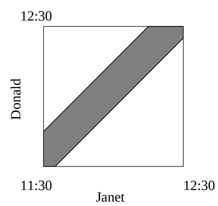
   We are given that $CD = 9$. One property of the centroid is that $\frac{CG}{GD} = \frac{AG}{GE} = \frac{BG}{GF} = 2$. So, $CG = 6$ and $GD = 3$. We are given that CD is perpendicular to another median, say AE. Let $a = GE$; then $AG = 2a$. By the Pythagorean Theorem, $a^2 + 36 = x^2$, $(2a)^2 + 36 = (2y)^2$, and $9 + (2a)^2 = 25$. The last tells us that $a = 2$, so then $x = 2\sqrt{10}$ and $y = \sqrt{13}$.

   Let $\theta = \angle BCA$. By the law of cosines with triangle ABC, $100 = (2y)^2 + (2x)^2 - 2(2x)(2y)\cos\theta$, so $xy\cos\theta = \frac{100-52-160}{-8} = 14$. By the law of cosines with the triangle BCF, $BF^2 = y^2 + (2x^2) - 2y(2x)\cos\theta = 13 + 160 - 4 \cdot 14 = 117$. So, $BF = 3\sqrt{13}$.

8. Janet and Donald agree to meet for lunch between 11:30 and 12:30. They each arrive at a random time in that interval. If Janet has to wait more than 15 minutes for Donald, she gets bored and leaves. Donald is busier so will only wait 5 minutes for Janet. What is the probability that the two will eat together? Express your answer as a fraction.

   Solution: $\mathbf{\frac{43}{144}}$

   We will solve this problem using a graph. Let the x-axis be time for Janet, and let the y-axis be time for Donald. Then, the region we are interested in is the square with both x and y between 11:30 and 12:30. Since Janet will wait no more than 15 minutes for Donald, we have $x \leq y + 15$. Since Donald will wait no more than 5 minutes for Janet, $y \leq x + 5$.



   The area of the whole square is 60^2, the area of the upper triangle is $\frac{45^2}{2}$, and the area of the lower triangle is $\frac{55^2}{2}$, so the probability that the two will eat together is $\frac{60^2 - \frac{45^2}{2} - \frac{55^2}{2}}{60^2} = \frac{43}{144}$.

9. What is the minimum number of straight cuts needed to cut a cake in 100 pieces? The pieces do not need to be the same size or shape but cannot be rearranged between cuts. You may assume that the cake is a large cube and may be cut from any direction.

   Solution: 9

   First, we will find the number of regions that n lines split a plane into. Notice that 0 lines split the plane into 1 region, 1 line splits the plane into 2 regions, 2 lines splits it into 4 regions, etc. In general, the k-th line can intersect each of the first $k-1$ lines exactly once, creating k more regions. So, the number of regions that n lines split a plane into is

$$1 + \sum_{k=1}^{n} k = \frac{n(n+1)}{2} + 1$$

   Now, cutting the cake is the same idea, only in 3 dimensions instead of a plane. Again, 0 planes split the cake into 1 piece, 1 plane gives 2 pieces, 2 planes give 4 pieces, etc. Notice that the k-th plane can intersect each of the first $k-1$ planes exactly once, but now the intersection is a line. Then, by the planar case, these lines divide the new plane into $\frac{(k-1)k}{2} + 1$ regions; each of these planar regions splits one piece of the cake into two, so we've added $\frac{(k-1)k}{2} + 1$ pieces. So, the number of regions that n lines split a cube into is

$$\begin{aligned} 1 + \sum_{k=1}^{n} \left(\frac{(k-1)k}{2} + 1 \right) &= 1 + \frac{1}{2}\sum_{k=1}^{n} k^2 - \frac{1}{2}\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\ &= 1 + \frac{1}{2}\frac{n(n+1)(2n+1)}{6} - \frac{1}{2}\frac{n(n+1)}{2} + n \\ &= n + 1 + \frac{2n^3 + 3n^2 + n}{12} - \frac{n^2 + n}{4} \\ &= \frac{2n^3 + 10n + 12}{12} = \frac{n^3 + 5n + 6}{6} \end{aligned}$$

   Then, the least n so that $\frac{n^3+5n+6}{6} \geq 100$ is 9.

10. You know that the binary function $\diamond$ takes in two non-negative integers and has the following properties:

- $0 \diamond a = 1$
- $a \diamond a = 0$
- If $a < b$, then $a \diamond b = (b - a)[(a - 1) \diamond (b - 1)]$.

Find a general formula for $x \diamond y$, assuming that $y \geq x > 0$.

Solution: $(y - x)^x$

We just need to check that this satisfies the three given properties. First, $0 \diamond a = (a - 0)^0 = 1$. Second, $a \diamond a = (a - a)^a = 0$. Finally, if $a < b$, then $a \diamond b = (b - a)^a = (b - a)[(b - a)^{a-1}] = (b - a)[(a - 1) \diamond (b - 1)]$.

11. Christopher and Robin are playing a game in which they take turns tossing a circular token of diameter 1 inch onto an infinite checkerboard whose squares have sides of 2 inches. If the token lands entirely in a square, the player who tossed the token gets 1 point; otherwise, the other player gets 1 point. A player wins as soon as he gets two more points than the other player. If Christopher tosses first, what is the probability that he will win? Express your answer as a fraction.

    Solution: $\frac{1}{2}$

    First, notice that the player tossing wins a point if the center of the coin lands in the center 1×1 by square of each 2×2 square of the checkerboard. So, the player tossing wins a point with probability $\frac{1}{4}$. Now, a player wins when the other player has n points and he has $n + 2$, making a total of $2n + 2$ tosses. We can think about the tosses in pairs; the first two are a pair, the second two are a pair, etc. The first player to win one of these pairs also wins the game, so the probability that Christopher wins the game in $2n + 2$ turns is the probability that, in the first n pairs, Christopher gets 1 point and Robin gets 1 point, and in the last pair, Christopher wins both points.

    Christopher tosses first in each pair, so his probability of winning the first toss is $\frac{1}{4}$, and his probability of winning the second toss by Robin not winning is $\frac{3}{4}$. So, the probability that he wins the first toss and loses the second toss is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. The probability that he loses the first toss and wins the second is $\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$. So, the probability that Christopher and Robin each get 1 point in the pair is $\frac{10}{16}$. The probability that this happens for the first n pairs is $\left(\frac{10}{16}\right)^n$. The probability that Christopher wins the last pair is $\frac{1}{4} \cdot \frac{3}{4}$. So, the probability that Christopher wins in $2n + 2$ turns is $\left(\frac{10}{16}\right)^n \frac{3}{16}$. This means that, overall, the probability that Christopher wins is

    $$\sum_{n=0}^{\infty} \left(\frac{10}{16}\right)^n \frac{3}{16} = \frac{3}{16}\left[1 + \frac{10}{16} + \left(\frac{10}{16}\right)^2 + \left(\frac{10}{16}\right)^3 + \ldots\right]$$

    This is a geometric series with sum

    $$\frac{3}{16} \cdot \frac{1}{1 - \frac{10}{16}} = \frac{1}{2}$$

    (Notice that the fact that the person tossing wins with probability $\frac{1}{4}$ is not important – no matter what that probability is, the overall probability of winning in such a game is still $\frac{1}{2}$!!)

12. A binary string is a string consisting of only 0's and 1's (for instance, 001010, 101, etc.). What is the probability that a randomly chosen binary string of length 10 has 2 consecutive 0's? Express your answer as a fraction.

    Solution: $\frac{55}{64}$

    We will first find the number of binary strings of length 10 that don't have 2 consecutive zeros. Let C_n be the set of binary strings of length n that don't have 2 consecutive zeros. Let A_n be the set of all strings in C_n that end in 1, and let B_n be the set of all strings in C_n that end in 0. Notice that $C_n = A_n \cup B_n$. Now, if a string is in A_{n+1}, then, the first n letters must be a string in C_n; conversely, appending a 1 to a string in C_n gives a string in A_{n+1}. So, $|A_{n+1}| = |C_n| = |A_n| + |B_n|$. Similarly, a string is in B_{n+1} iff the first n letters are in A_n, so $|B_{n+1}| = |A_n|$. Thus, $|A_{n+2}| = |A_{n+1}| + |B_{n+1}| = |A_{n+1}| + |A_n|$, which gives us a Fibonacci sequence which starts with $|A_1| = 1$, $|A_2| = 2$. It's easy to find that $|A_{10}| = 89$ and $|B_{10}| = 55$, so $|C_{10}| = 89 + 55 = 144$. Therefore, there are $2^{10} - 144 = 1024 - 144 = 880$ strings without 2 consecutive 0's. So, the probability that a randomly chosen string of length 10 has 2 consecutive 0's is $\frac{880}{1024} = \frac{55}{64}$.

13. You have 2 six-sided dice. One is a normal fair die, while the other has 2 ones, 2 threes, and 2 fives. You pick a die and roll it. Because of some secret magnetic attraction of the unfair die, you have a 75% chance of picking the unfair die and a 25% chance of picking the fair die. If you roll a three, what is the probability that you chose the fair die?

    Solution: $\frac{1}{7}$

    The probability of picking the fair die is $\frac{1}{4}$, and the probability of rolling a 3 with it is $\frac{1}{6}$. So, the probability of picking the fair die and rolling a 3 with it is $\frac{1}{24}$. The probability of picking the unfair die and rolling a 3 with it is $\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$. So, the probability that we chose the fair die is

$$\frac{\frac{1}{24}}{\frac{1}{24} + \frac{1}{4}} = \frac{1}{7}$$

14. Find the prime factorization of $\sum_{1 \le i < j \le 100} ij$.

    Solution: $\mathbf{3 \cdot 5^2 \cdot 11 \cdot 101 \cdot 151}$

    We will find $\sum_{1 \le i < j \le n} ij$ for any n. Since $(1 + 2 + \ldots + n) = \frac{n(n+1)}{2}$, we square and get that $(1 + 2 + \ldots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2$. Since $2\sum_{1 \le i < j \le n} ij = (1 + 2 + \ldots + n)^2 - (1^2 + 2^2 + \ldots + n^2)$, we have

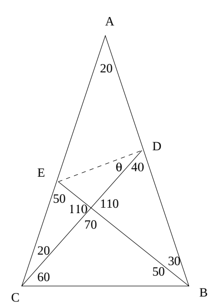
$$\begin{aligned}
\sum_{1 \le i < j \le n} ij &= \frac{1}{2}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{3n^4 + 6n^3 + 3n^2 - 2(2n^3 + 3n^2 + n)}{24} \\
&= \frac{3n^4 + 2n^3 - 3n^2 - 2n}{24} = \frac{n(3n^3 + 2n^2 - 3n - 2)}{24} \\
&= \frac{n(3n+2)(n^2-1)}{24} = \frac{(n-1)n(n+1)(3n+2)}{24}
\end{aligned}$$

    So, for $n = 100$, we find that the sum is $\frac{99 \cdot 100 \cdot 101 \cdot 302}{24} = 33 \cdot 25 \cdot 101 \cdot 151 = 3 \cdot 5^2 \cdot 11 \cdot 101 \cdot 151$.

15. Let ABC be an isosceles triangle with $\angle ABC = \angle ACB = 80°$. Let D be a point on AB such that $\angle DCB = 60°$ and E be a point on AC such that $\angle ABE = 30°$. Find $\angle CDE$ in degrees.

    Solution: 30°

    Let $\theta = \angle CDE$.



Since we're only worried about angles in this problem, we can just pretend that $AD = 1$. Then, since ADC is an isosceles triangle with side 1 and base angle $20°$, $AC = 2\cos 20°$. Since ABC is an isosceles triangle with vertex angle $20°$ and side $2\cos 20°$, $BC = 4\cos 20° \sin 10°$. Notice that ECB is isosceles, so $EC = BC = 4\cos 20° \sin 10°$. Also, ADC is isosceles, so $DC = AD = 1$.

Now, let's look at the triangle EDC. Notice that $\angle DEC = 110 - \theta$. By the law of sines,

$$\frac{\sin\theta}{EC} = \frac{\sin(160° - \theta)}{DC}$$

so

$$\begin{aligned}
\frac{\sin\theta}{4\cos 20^\circ \sin 10^\circ} &= \sin(160^\circ - \theta) \\
&= \sin(20^\circ + \theta) \\
&= \sin 20^\circ \cos\theta + \cos 20^\circ \sin\theta
\end{aligned}$$

Then,

$$\cos\theta \sin 20^\circ = \sin\theta \left(\frac{1}{4\cos 20^\circ \sin 10^\circ} - \cos 20^\circ \right) = \sin\theta \left(\frac{1 - 4\cos^2 20^\circ \sin 10^\circ}{4\cos 20^\circ \sin 10^\circ} \right)$$

So,

$$\begin{aligned}
\frac{1}{\tan\theta} &= \frac{1 - 4\cos^2 20^\circ \sin 10^\circ}{4\cos 20^\circ \sin 10^\circ \sin 20^\circ} \\
&= \frac{1 - 4\cos^2 20^\circ \sin 10^\circ}{2\sin 40^\circ \sin 10^\circ} \text{ since } \sin(2\phi) = 2\sin\phi\cos\phi \\
&= \frac{\cos 10^\circ - 4\cos^2 20^\circ \sin 10^\circ \cos 10^\circ}{2\sin 40^\circ \sin 10^\circ \cos 10^\circ} \\
&= \frac{\cos 10^\circ - 2\cos^2 20^\circ \sin 20^\circ}{\sin 40^\circ \sin 20^\circ} \\
&= \frac{\sin 80^\circ - \cos 20^\circ \sin 40^\circ}{\sin 40^\circ \sin 20^\circ} = \frac{2\sin 40^\circ \cos 40^\circ - \cos 20^\circ \sin 40^\circ}{\sin 40^\circ \sin 20^\circ} \\
&= \frac{2\cos 40^\circ - \cos 20^\circ}{\sin 20^\circ} = \frac{2\cos(30^\circ + 10^\circ) - \cos(30^\circ - 10^\circ)}{\sin(30^\circ - 10^\circ)} \\
&= \frac{2(\cos 30^\circ \cos 10^\circ - \sin 30^\circ \sin 10^\circ) - (\cos 30^\circ \cos 10^\circ + \sin 30^\circ \sin 10^\circ)}{\sin 30^\circ \cos 10^\circ - \cos 30^\circ \sin 10^\circ} \\
&= \frac{\frac{\sqrt{3}}{2}\cos 10^\circ - \frac{3}{2}\sin 10^\circ}{\frac{1}{2}\cos 10^\circ - \frac{\sqrt{3}}{2}\sin 10^\circ} = \sqrt{3}
\end{aligned}$$

So, $\tan\theta = \frac{\sqrt{3}}{3}$, which means that $\theta = 30^\circ$.