2001 Stanford Math Tournament
Algebra

1. Find the result of adding seven to the result of forty divided by one-half.

2. Each valve A, B, and C, when open, releases water into a tank at its own constant rate. With all three valves open, the tank fills in 1 hour, with only valves A and C open it takes 1.5 hours, and with only valves B and C open it takes 2 hours. How many hours will it take to fill the tank with only valves A and B open?

3. Julie has a 12 foot by 20 foot garden. She wants to put fencing around it to keep out the neighbor's dog. Normal fenceposts cost \$2 each while strong ones cost \$3 each. If Julie needs one fencepost for every 2 feet and has \$70 to spend on fenceposts, what is the greatest number of strong fenceposts she can buy?

4. $p(x)$ is a real polynomial of degree at most 3. Suppose there are four distinct solutions to the equation $p(x) = 7$. What is $p(0)$?

5. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = \begin{cases} 2, & x = 0 \\ (f(x-1))^2, & x \neq 0 \end{cases}$ What is $\log_2 f(11)$?

6. If for three distinct positive numbers x, y, and z,
$$\frac{y}{x-z} = \frac{x+y}{z} = \frac{x}{y},$$
then find the numerical value of x/y.

7. If $\log_A B + \log_B A = 3$ and $A < B$, find $\log_B A$.

8. Determine the value of $1 + \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{1+\cdots}}}}$.

9. Find all solutions to $(x-3)(x-1)(x+3)(x+5) = 13$.

10. Suppose x, y, z satisfy
$$\begin{aligned} x + y + z &= 3 \\ x^2 + y^2 + z^2 &= 5 \\ x^3 + y^3 + z^3 &= 7 \end{aligned}$$
Find $x^4 + y^4 + z^4$.

2001 Stanford Math Tournament
Advanced Topics

1. Suppose $\sinh(x) = \frac{e^x - e^{-x}}{2}$. What is the inverse function of $\sinh(x)$?

2. Write $20_{10}\frac{70_{17}!}{170_8!}$ as a decimal in base 6. The subscript indicates the base in which the number is written (i.e., 20_{10} is 20 base 10.)

3. There are 36 penguins in a row, and Barbara Manatee is standing in front of them. In general, a penguin rotation of penguins $p_1, p_2, \ldots, p_n$ is a rearrangement of them such that p_1 moves to where p_2 was standing, and in general p_i moves to where p_{i+1} was standing, and p_n moves to where p_1 was standing. So, after a penguin rotation, the new order of these penguins is $p_n, p_1, p_2, \ldots, p_{n-1}$. Whenever Barbara Manatee blows her whistle, the 2-4 penguins go through a penguin rotation, the 5-9 penguins go through a penguin rotation, the 10-16 penguins go through a penguin rotation, the 17-25 penguins go through a penguin rotation, and the 26-36 penguins go through a penguin rotation. What is the least positive number of whistle blows such that the penguins all return to their original position?

4. Eleven pirates find a treasure chest. When they split up the coins in it, evenly among all the pirates, they find that there are 5 coins left. They throw one pirate overboard and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split perfectly. What is the least number of coins there could have been?

5. Evaluate $1 \cdot 2^{-1} + 3 \cdot 2^{-2} + 5 \cdot 2^{-3} + 7 \cdot 2^{-4} + \ldots$

6. How many subsets of $\{n \mid n > 0$ and n is a multiple of 3 less than $100\}$ are also subsets of $\{n \mid n > 0$ and n is a multiple of 4 less than $100\}$?

7. There are 2000 dots spaced evenly around a circle. If 4 distinct dots A, B, C, and D are picked randomly, what is the probability that AB intersects CD?

8. Ashley, Bob, Carol, and Doug are rescued from a desert island by a pirate who forces them to play a game. Each of the four, in alphabetical order by their first names, is forced to roll two dice. If the total on the two dice is either 8 or 9, the person rolling the dice is forced to walk the plank. The players go in order until one player loses: A, B, C, D, A, B, ... What is the probability that Doug survives?

9. There are 19 men numbered 1 though 19 and 20 women numbered 1 through 20 entered in a computer dating service. The computer wants to match every man to a compatible woman, and each man is only compatible with women who have a number that is greater than equal to his (i.e. man 19 is only compatible with women 19 and 20, man 18 is only compatible with women 18, 19, 20, etc.). If each women is matched with at most one man, let n be the number of ways that the computer can match them. What is the prime factorization of n?

10. David is playing with Legos with velcro attached to the ends. He has green Legos of length 1, blue Legos of length 2, and red Legos of length 3, and wants to combine them (by attaching them at the ends) to make a "super-lego" of length 10. If any different ordering of colors is considered a distinct "super-lego", how many ways can he make this "super-lego"?

2001 Stanford Math Tournament
Calculus

1. What is $\frac{d^{2959}}{dx^{2959}} \sin x$?

2. If $f(x) = [x]$ is the greatest integer function, what is $f'''(3.7)$?

3. Suppose that f is a monotonically increasing continuous function defined on the real numbers. We know that $f(0) = 0$ and $f(2) = 3$. Let S be the set of all possible values of $\int_0^2 f(x)\,dx$. What is the least upper bound of S?

4. Evaluate $\int_{-4}^{5} \frac{x^2}{|x|} dx$.

5. $\int_1^{\infty} \frac{1}{x^2+3x} =$?

6. Given a point (p, q) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $p \neq 0$, find the x-intercept of the tangent line at (p, q) in terms of p, a, and b. (Note that $a, b \neq 0$.)

7. Find the number of real solutions to $\sin(6\pi x) = x$, where x is in radians.

8. If a, b, and c are positive real numbers such that $a + b + c = 16$ and $a^2 + b^2 + c^2 = 160$, what is the maximum possible value of abc?

9. For a given $k > 0, n \geq 2k > 0$, consider the square R in the plane consisting of all points (x, y) with $0 \leq x, y \leq n$. Color each point in R gray if $\frac{xy}{k} \leq x + y$, and blue otherwise. Find the area of the gray region in terms of n and k.

10. Let $f(x) = \left(x^2 - 1\right)^n$, where n is a positive integer. Determine, in terms of n, (a, b, c), where a,b, and c are the number of distinct roots of $f^{(n)}(x)$ in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, +\infty)$, respectively.

2001 Stanford Math Tournament
Geometry

1. Find the coordinates of the points of intersection of the graphs of the equations $y = |2x| - 2$ and $y = -|2x| + 2$.

2. Jacques is building an igloo for his dog. The igloo's inside and outside are both perfectly hemispherical. The interior height at the center is 2 feet. The igloo has no door yet and contains $\frac{254}{2187}\pi$ cubic yards of hand-packed snow. What is the circumference of the igloo at its base in feet?

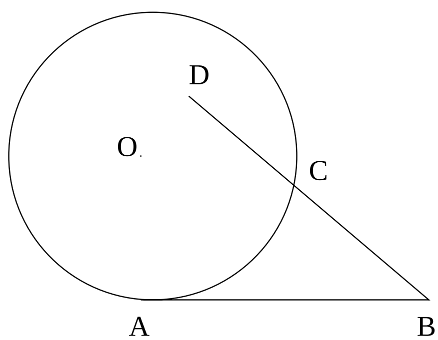
3. Find the area of the convex quadrilateral whose vertices are $(0,0), (4,5), (9,21), (-3,7)$.

4. E is a point in the interior of rectangle $ABCD$. $AB = 6$, triangle ABE has area 6, and triangle CDE has area 12. Find $(EA)^2 - (EB)^2 + (EC)^2 - (ED)^2$.

5. Two identical cones, each 2 inches in height, are held one directly above another with the pointed end facing down. The upper cone is completely filled with water. A small hole is punctured in the bottom of the upper cone so that the water trickles down into the bottom cone. When the water reaches a depth of 1 inch in the bottom cone, what is its depth in the upper cone?

6. Find the radius of a circle inscribed in the triangle determined by the lines $4x + 3y = 24$, $56x - 33y = -264$, and $3x - 4y = 18$.

7. In the figure, AB is tangent at A to the circle with center O; point D is interior to the circle; and DB intersects the circle at C. If $BC = DC = 3$, $OD = 2$ and $AB = 6$, then find the radius of the circle.



8. Let S be the solid tetrahedron with boundary points $(0,0,0), (2,4,0), (5,1,0), (3,2,10)$. Let $z_1 = \max\{q \mid (\frac{12}{5}, \frac{23}{10}, q) \in S\}$ and let $z_2 = \max\{r \mid (\frac{9}{2}, \frac{5}{4}, r) \in S\}$. Find $z_1 - z_2$.

9. Circles A and B are tangent and have radii 1 and 2, respectively. A tangent to circle A from the point B intersects circle A at C. D is chosen on circle B so that AC is parallel to BD and the two segments BC and AD do not intersect. Segment AD intersects circle A at E. The line through B and E intersects circle A through another point F. Find EF.

10. E is a point inside square $ABCD$ such that $\angle ECD = \angle EDC = 15°$. Find $\angle AEB$.

2001 Stanford Math Tournament
General Test

1. If $a \star b$ is defined as $2a - b^a$, what value is associated with $3 \star 2$?

2. Find the coordinates of the points of intersection of the graphs of the equations $y = |2x| - 2$ and $y = -|2x| + 2$.

3. ABCD is a 4-digit number. What is the largest number that can be formed with AB a prime 2-digit number and C and D each a prime 1-digit number?

4. Let A be the set of all non-composite positive integers. Let B be the set of all squares of integers. Let C be the set of all multiples of 3. What is the size of $A \cap (B \cup C)$?

5. What is i^{2959}?

6. The points $Q = (9, 14)$ and $R = (a, b)$ are symmetric with respect to the point $P = (5, 3)$. What are the coordinates of point R?

7. If $F(x) = 3x^3 - 2x^2 + x - 3$, find $F(1 + i)$.

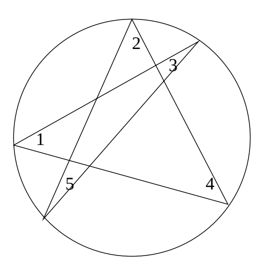
8. Express the absolute value of the difference between $0.\overline{36}$ and 0.36 as a common fraction.

9. Jonathan chooses 10 cards without replacement from a standard 52-card deck of cards (without jokers). What is the probability that he does <u>not</u> draw the 3 of clubs <u>and</u> he does <u>not</u> draw the King of spades?

10. Julie has a 12 foot by 20 foot garden. She wants to put fencing around it to keep out the neighbor's dog. Normal fenceposts cost \$2 each while strong ones cost \$3 each. If Julie needs one fencepost for every 2 feet and has \$70 to spend on fenceposts, what is the largest number of strong fenceposts she can buy?

11. Anne has a cube that she wants to paint. If she decides to paint each face a different color and she has 6 colors, how many distinct ways can she paint the cube? (If one painted cube can be rotated to get another, they are the same.)

12. Find, in degrees, the sum of angles 1, 2, 3, 4, and 5 in the star-shaped figure shown.



13. Each valve A, B, and C, when open, releases water into a tank at its own constant rate. With all three valves open, the tank fills in 1 hour, with only valves A and C open it takes 1.5 hours, and with only valves B and C open it takes 2 hours. How many hours will it take to fill the tank with only valves A and B open?

14. Lattice paths are paths consisting of one unit steps in the positive horizontal or vertical directions. How many distinct lattice paths are there from the origin to the point $(5, 4)$?

15. Three circles, each of area 4π, are all externally tangent. Their centers form a triangle. What is the area of the triangle?

16. Two equilateral triangles sharing an edge have a combined area of π. What is the square of the length of their shared edge?

17. Frogger wants to cross a stream. He starts on one bank and jumps onto the first passing log, which is traveling at 9 feet per second. The following logs are at speeds such that the second is twice is fast as the first, the third is twice as fast as the second, the fourth is one-third as fast as the third, and the last (fifth) is one-third as fast as the fourth (all of the logs travel in the same direction). If there is a 2 second interval between jumps, then how far does he travel down river from his original point, once he reaches the opposite bank?

18. Rice tuition was \$7200 one semester. Each student takes a minimum of 12 credits and a maximum of 20 credits. For each credit, a class meets for 3 hours a week. A semester is 15 weeks with no breaks. What is the difference in cost in dollars per hour of class that a student taking the maximum load and a student taking the minimum load pays, to the nearest penny?

19. Suppose $a_1, a_2, a_3, \ldots$ is a sequence of numbers such that $a_1 = 1$ and $a_{n+1} = a_n + (2n+1)$ for all positive integers n. Find a_{20}.

20. My spouse and I have 9 kids. Each child gets married and has exactly 9 children of their own. By the time I have 100,000 descendants (including every generation) what is the longest title that applies to me? (Assuming no married blood relatives.)

    a. grandparent
    b. great grandparent
    c. great great grandparent
    d. great great great grandparent
    e. great great great great grandparent
    f. great great great great great grandparent
    g. great great great great great great grandparent

21. Find the greatest integer x for which $3^{20} > 32^x$.

22. If $x + y = xy$, with x and y real, what value can x not have?

23. Instead of using two standard cubical dice in a board game, three standard cubical dice are used so that the game goes more quickly. In the regular game, doubles are needed to get out of the "pit". In the revised game, doubles or triples will get you out. How many times as likely is it for a player to get out of the "pit" on one toss under the new rules as compared to the old rules?

24. A circle is drawn with center at the origin and radius 2.5. Find the coordinates of all intersections of the circle with an origin-centered square of side length 4 whose sides are parallel to the coordinate axes.

25. Write $20_{10}\frac{(70_{17})!}{(170_8)!}$ as a decimal in base 6. The subscript indicates the base in which the number is written (i.e., 15_6 is 15 base 6, so $15_6 = 11_{10}$)

26. Jacques is building an igloo for his dog. The igloo's inside and outside are both perfectly hemispherical. The interior height at the center is 2 feet. The igloo has no door yet and contains $\frac{254}{2187}\pi$ cubic yards of hand-packed snow. What is the circumference of the igloo at its base in feet?

27. How many solutions are there to $x^7 + y^7 = z^7$ with x, y, z real?

28. Find all prime factors of $3^{18} - 2^{18}$.

29. If $a \neq 1$ and $\sqrt[a]{10000_a} = 10_a$, find a. The subscript indicates the base in which the number is written.

30. Suppose 100 students have at least one of math, applied math, or statistics as one of their majors. There are 24 statistics majors, 46 applied math majors, and 55 pure math majors. There are 14 who are at least pure & applied math, 10 at least applied math & statistics and 7 at least statistics & pure math. How many triple majors (people with all three majors) are there, if any?

31. The sum of 3 real numbers is 0. If the sum of their cubes is π^e, what is their product?

32. Given a real-valued function $f(x) = \sqrt{\frac{x+3}{|2x+6|}}$, what is the least integer value which lies in the domain of the function?

33. E is a point in the interior of rectangle $ABCD$. $AB = 6$, triangle ABE has area 6, and triangle CDE has area 12. Find $(EA)^2 - (EB)^2 + (EC)^2 - (ED)^2$.

34. What is the radius of the incircle of a right triangle with legs of lengths 7 and 24?

35. If $\log_A B + \log_B A = 3$ and $A < B$, find $\log_B A$.

36. ABC is an equilateral triangle with edge of length 256. Let n be the maximum number of non-overlapping equilateral triangles of edge length $\frac{1}{2}$ that can be fit into ABC. What is the prime factorization of n?

37. If $x^2 + \frac{1}{x^2} = 7$, find $\left|x^3 + \frac{1}{x^3}\right|$.

38. Eleven pirates find a treasure chest. When they split up the coins in it evenly among all the pirates, they find that there are 5 coins left. They throw one pirate overboard and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split perfectly. What is the least number of coins there could have been?

2001 Stanford Math Tournament
Team Test

1. $ABCD$ is a square with sides of unit length. Points E and F are taken on sides AB and AD respectively so that $AE = AF$ and the quadrilateral $CDFE$ has maximum area. What is this maximum area?

2. How many positive integers between 1 and 400 (inclusive) have exactly 15 positive integer factors?

3. Find the 2000th positive integer that is not the difference between any two integer squares.

4. For what values of a does the system of equations

$$x^2 = y^2, (x-a)^2 + y^2 = 1$$

have exactly 2 solutions?

5. What quadratic polynomial whose coefficient of x^2 is 1 has roots which are the complex conjugates of the solutions of $x^2 - 6x + 11 = 2xi - 10i$? (Note that the complex conjugate of $a + bi$ is $a - bi$, where a and b are real numbers.)

6. Find the least n such that any subset of $\{1, 2, \ldots, 100\}$ with n elements has 2 elements with a difference of 9.

7. The median to a 10 cm side of a triangle has length 9 cm and is perpendicular to a second median of the triangle. Find the exact value in centimeters of the length of the third median.

8. Janet and Donald agree to meet for lunch between 11:30 and 12:30. They each arrive at a random time in that interval. If Janet has to wait more than 15 minutes for Donald, she gets bored and leaves. Donald is busier so will only wait 5 minutes for Janet. What is the probability that the two will eat together? Express your answer as a fraction.

9. What is the minimum number of straight cuts needed to cut a cake in 100 pieces? The pieces do not need to be the same size or shape but cannot be rearranged between cuts. You may assume that the cake is a large cube and may be cut from any direction.

10. You know that the binary function $\diamond$ takes in two non-negative integers and has the following properties:

    - $0 \diamond a = 1$
    - $a \diamond a = 0$
    - If $a < b$, then $a \diamond b = (b - a)[(a - 1) \diamond (b - 1)]$.

    Find a general formula for $x \diamond y$, assuming that $y \geq x > 0$.

11. Christopher and Robin are playing a game in which they take turns tossing a circular token of diameter 1 inch onto an infinite checkerboard whose squares have sides of 2 inches. If the token lands entirely in a square, the player who tossed the token gets 1 point; otherwise, the other player gets 1 point. A player wins as soon as he gets two more points than the other player. If Christopher tosses first, what is the probability that he will win? Express your answer as a fraction.

12. A binary string is a string consisting of only 0's and 1's (for instance, 001010, 101, etc.). What is the probability that a randomly chosen binary string of length 10 has 2 consecutive 0's? Express your answer as a fraction.

13. You have 2 six-sided dice. One is a normal fair die, while the other has 2 ones, 2 threes, and 2 fives. You pick a die and roll it. Because of some secret magnetic attraction of the unfair die, you have a 75% chance of picking the unfair die and a 25% chance of picking the fair die. If you roll a three, what is the probability that you chose the fair die?

14. Find the prime factorization of $\sum_{1 \leq i < j \leq 100} ij$.

15. Let ABC be an isosceles triangle with $\angle ABC = \angle ACB = 80°$. Let D be a point on AB such that $\angle DCB = 60°$ and E be a point on AC such that $\angle ABE = 30°$. Find $\angle CDE$ in degrees.