Advanced Topics
Stanford Mathematics Tournament 2000

1. How many different ways are there to paint the sides of a tetrahedron with exactly 4 colors? Each side gets its own color, and two colorings are the same if one can be rotated to get the other.

2. Simplify $\left(\frac{-1+i\sqrt{3}}{2}\right)^6 + \left(\frac{-1-i\sqrt{3}}{2}\right)^6$ to the form $a + bi$.

3. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$.

4. Five positive integers from 1 to 15 are chosen without replacement. What is the probability that their sum is divisible by 3?

5. Find all 3-digit numbers which are the sums of the cubes of their digits.

6. 6 people each have a hat. If they shuffle their hats and redistribute them, what is the probability that exactly one person gets their own hat back?

7. Assume that a, b, c, d are positive integers, and $\frac{a}{c} = \frac{b}{d} = \frac{3}{4}$, $\sqrt{a^2 + c^2} - \sqrt{b^2 + d^2} = 15$. Find $ac + bd - ad - bc$.

8. How many non-isomorphic graphs with 9 vertices, with each vertex connected to exactly 6 other vertices, are there? (Two graphs are isomorphic if one can relabel the vertices of one graph to make all edges be exactly the same.)

9. The Cincinnati Reals are playing the Houston Alphas in the last game of the Swirled Series. The Alphas are leading by 1 run in the bottom of the 9th (last) inning, and the Reals are at bat. Each batter has a $\frac{1}{3}$ chance of hitting a single and a $\frac{2}{3}$ chance of making an out. If the Reals hit 5 or more singles before they make 3 outs, they will win. If the Reals hit exactly 4 singles before making 3 outs, they will tie the game and send it into extra innings, and they will have a $\frac{3}{5}$ chance of eventually winning the game (since they have the added momentum of coming from behind). If the Reals hit fewer than 4 singles, they will LOSE! What is the probability that the Alphas hold off the Reals and win, sending the packed Alphadome into a frenzy? Express the answer as a fraction.

10. I call two people A and B and think of a natural number n. Then I give the number n to A and the number $n+1$ to B. I tell them that they have both been given natural numbers, and further that they are consecutive natural numbers. However, I don't tell A what B's number is and vice versa. I start by asking A if he knows B's number. He says "no". Then I ask B if he knows A's number, and he says "no" too. I go back to A and ask, and so on. A and B can both hear each other's responses. Do I ever get a "yes" in response? If so, who responds first with "yes" and how many times does he say "no" before this? Assume that both A and B are very intelligent and logical. You may need to consider multiple cases.

Algebra
Stanford Mathematics Tournament 2000

1. How many integers x satisfy $|x| + 5 < 7$ and $|x - 3| > 2$?

2. Evaluate $2000^3 - 1999 \cdot 2000^2 - 1999^2 \cdot 2000 + 1999^3$.

3. Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores?

4. What is the fewest number of multiplications required to reach x^{2000} from x, using only previously generated powers of x? For example, $x \to x^2 \to x^4 \to x^8 \to x^{16} \to x^{32} \to x^{64} \to x^{128} \to x^{256} \to x^{512} \to x^{1024} \to x^{1536} \to x^{1792} \to x^{1920} \to x^{1984} \to x^{2000}$ uses 15 multiplications.

5. A jacket was originally priced \$100. The price was reduced by 10% three times and increased by 10% four times in some order. To the nearest cent, what was the final price?

6. Barbara, Edward, Abhinav, and Alex took turns writing this test. Working alone, they could finish it in 10, 9, 11, and 12 days, respectively. If only one person works on the test per day, and nobody works on it unless everyone else has spent at least as many days working on it, how many days (an integer) did it take to write this test?

7. A number n is called multiplicatively perfect if the product of all the positive divisors of n is n^2. Determine the number of positive multiplicatively perfect numbers less than 100.

8. A man has three daughters. The product of their ages is 168, and he remembers that the sum of their ages is the number of trees in his yard. He counts the trees but cannot determine any of their ages. What are all possible ages of his oldest daughter?

9. $\frac{a}{c} = \frac{b}{d} = \frac{3}{4}$, $\sqrt{a^2 + c^2} - \sqrt{b^2 + d^2} = 15$. Find $ac + bd - ad - bc$.

10. Find the smallest positive integer a such that $x^4 + a^2$ is not prime for any integer x.

Calculus
Stanford Mathematics Tournament 2000

1. Find the slope of the tangent at the point of inflection of $y = x^3 - 3x^2 + 6x + 2000$.

2. Karen is attempting to climb a rope that is not securely fastened. If she pulls herself up x feet at once, then the rope slips x^3 feet down. How many feet at a time must she pull herself up to climb with as few pulls as possible?

3. A rectangle of length $\frac{1}{4}\pi$ and height 4 is bisected by the x-axis and is in the first and fourth quadrants, with the leftmost edge on the y-axis. The graph of $y = \sin(x) + C$ divides the area of the square in half. What is C?

4. For what value of x $(0 < x < \frac{\pi}{2})$ does $\tan x + \cot x$ achieve its minimum?

5. For $-1 < x < 1$, let $f(x) = \sum_{i=1}^{\infty} \frac{x^i}{i}$. Find a closed form expression (a closed form expression is one not involving summation) for f.

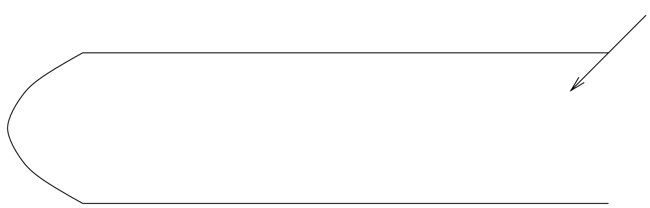
6. A hallway of width 6 feet meets a hallway of width $6\sqrt{5}$ feet at right angles. Find the length of the longest pipe that can be carried horizontally around this corner.

7. An **envelope** of a set of lines is a curve tangent to all of them. What is the envelope of the family of lines $y = \frac{2}{x_0} + x(1 - \frac{1}{x_0^2})$, with x_0 ranging over the positive real numbers?

8. Find $\int_0^{\frac{\pi}{2}} \ln \sin \theta d\theta$.

9. Let $f(x) = \sqrt{x + \sqrt{0 + \sqrt{x + \sqrt{0 + \sqrt{x + \ldots}}}}}$. If $f(a) = 4$, then find $f'(a)$.

10. A mirror is constructed in the shape of y equals $\pm\sqrt{x}$ for $0 \leq x \leq 1$, and ± 1 for $1 < x < 9$. A ray of light enters at (10,1) with slope 1. How many times does it bounce before leaving?



General
Stanford Mathematics Tournament 2000

1. If $a = 2b + c$, $b = 2c + d$, $2c = d + a - 1$, $d = a - c$, what is b?

2. The temperatures $f°$F and $c°$C are equal when $f = \frac{9}{5}c + 32$. What temperature is the same in both °F and °C?

3. A twelve foot tree casts a five foot shadow. How long is Henry's shadow (at the same time of day) if he is five and a half feet tall?

4. Tickets for the football game are \$10 for students and \$15 for non-students. If 3000 fans attend and pay \$36250, how many students went?

5. Find the interior angle between two sides of a regular octagon (degrees).

6. Three cards, only one of which is an ace, are placed face down on a table. You select one, but do not look at it. The dealer turns over one of the other cards, which is not the ace (if neither are, he picks one of them randomly to turn over). You get a chance to change your choice and pick either of the remaining two face-down cards. If you selected the cards so as to maximize the chance of finding the ace on the second try, what is the probability that you selected it on the

   (a) first try?

   (b) second try?

7. Find $[\sqrt{19992000}]$ where $[x]$ is the greatest integer less than or equal to x.

8. Bobo the clown was juggling his spherical cows again when he realized that when he drops a cow is related to how many cows he started off juggling. If he juggles 1, he drops it after 64 seconds. When juggling 2, he drops one after 55 seconds, and the other 55 seconds later. In fact, he was able to create the following table:

cows started juggling	1	2	3	4	5	6	7	8	9	10	11
seconds he drops after	64	55	47	40	33	27	22	18	14	13	12
cows started juggling	12	13	14	15	16	17	18	19	20	21	22
seconds he drops after	11	10	9	8	7	6	5	4	3	2	1

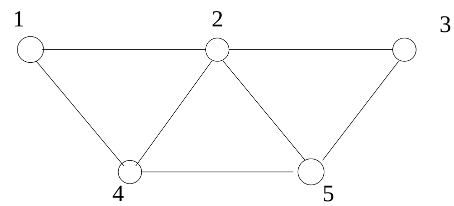
   He can only juggle up to 22 cows. To juggle the cows the longest, what number of cows should he start off juggling? How long (in minutes) can he juggle for?

9. Edward's formula for the stock market predicts correctly that the price of HMMT is directly proportional to a secret quantity x and inversely proportional to y, the number of hours he slept the night before. If the price of HMMT is \$12 when $x = 8$ and $y = 4$, how many dollars does it cost when $x = 4$ and $y = 8$?

10. Bob has a 12 foot by 20 foot garden. He wants to put fencing around it to keep out the neighbor's dog. Normal fenceposts cost \$2 each while strong ones cost \$3 each. If he needs one fencepost for every 2 feet and has \$70 to spend on the fenceposts, what is the largest number of strong fenceposts he can buy?

11. If $a@b = \frac{a+b}{a-b}$, find n such that $3@n = 3$.

12. In 2020, the United States admits North Mathematica as the 51st state. It consists of 5 islands joined by bridges as shown. Is it possible to cross all the bridges without doubling over? If so, what is the difference (positive) between the number of the start island and the number of the end island?



13. How many permutations of 123456 have exactly one number in the correct place?

14. The author of this question was born on April 24, 1977. What day of the week was that?

15. Which is greater: $(3^5)^{(5^3)}$ or $(5^3)^{(3^5)}$?

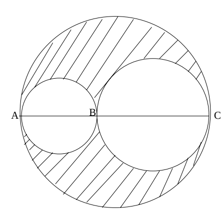
16. Joe bikes x miles East at 20 mph to his friend's house. He then turns South and bikes x miles at 20 mph to the store. Then, Joe turns East again and goes to his grandma's house at 14 mph. On this last leg, he has to carry flour he bought for her at the store. Her house is 2 more miles from the store than Joe's friend's house is from the store. Joe spends a total of 1 hour on the bike to get to his grandma's house. If Joe then rides straight home in his grandma's helicopter at 78 mph, how many minutes does it take Joe to get home from his grandma's house?

17. In how many distinct ways can the letters of STANTON be arranged?

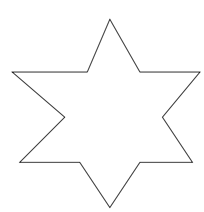
18. You use a lock with four dials, each of which is set to a number between 0 and 9 (inclusive). You can never remember your code, so normally you just leave the lock with each dial one higher than the correct value. Unfortunately, last night someone changed all the values to 5. All you remember about your code is that none of the digits are prime, 0, or 1, and that the average value of the digits is 5. How many combinations will you have to try?

19. Eleven pirates find a treasure chest. When they split up the coins in it, they find that there are 5 coins left. They throw one pirate overboard and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split evenly. What is the least number of coins there could have been?

20. Given: AC has length 5, semicircle AB has radius 1, semicircle BC has diameter 3. What percent of the the big circle is shaded?



21. Find the area of the six-pointed star if all edges are of length s, all acute angles are $60°$ and all obtuse angles are $240°$.



22. An equilateral triangle with sides of length 4 has an isosceles triangle with the same base and half the height cut out of it. Find the remaining area.

23. What are the last two digits of $7^{7^{7^{7}}}$?

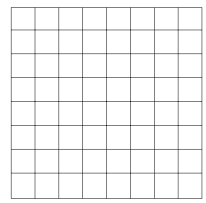
24. Peter is randomly filling boxes with candy. If he has 10 pieces of candy and 5 boxes in a row labeled A, B, C, D, and E, how many ways can he distribute the candy so that no two adjacent boxes are empty?

25. How many points does one have to place on a unit square to guarantee that two of them are strictly less than 1/2 unit apart?

26. Janet is trying to find Tim in a Cartesian forest. Janet is $5\sqrt{2}$ miles from (0,0), $\sqrt{41}$ miles from (1,0), and $\sqrt{61}$ miles from (0,1). Tim is $\sqrt{65}$ miles from (0,0), $2\sqrt{13}$ miles from (1,0), and $\sqrt{58}$ miles from (0,1). How many miles apart are Janet and Tim?

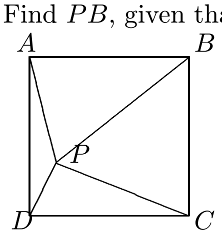
Geometry
Stanford Mathematics Tournament 2000

1. How many rectangles are there on an $8x8$ checkerboard?

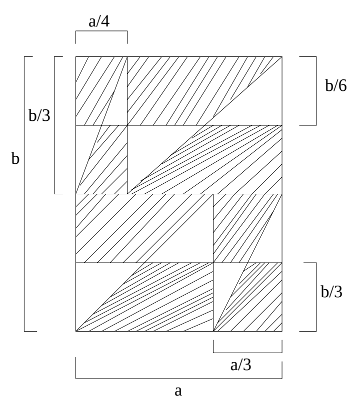


2. In a triangle the sum of squares of the sides is 96. What is the maximum possible value of the sum of the medians?

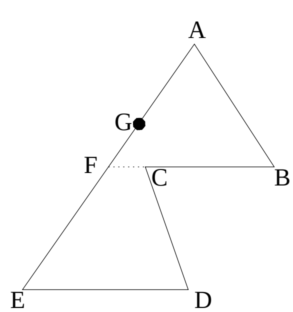
3. Find PB, given that $PA = 15, PC = 20, PD = 7$, and $ABCD$ is a square.



4. Find the total area of the non-triangle regions in the figure below (the shaded area).



5. Side $\overline{AB} = 3$. $\triangle ABF$ is an equilateral triangle. Side $\overline{DE} = \overline{AB} = \overline{AF} = \overline{GE}$. $\angle FED = 60°$. $\overline{FG} = 1$. Calculate the area of $ABCDE$.

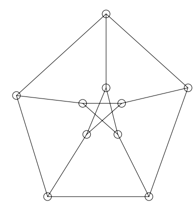


6. What is the area of the largest circle contained in an equilateral triangle of area $8\sqrt{3}$?

7. Let ABC be a triangle inscribed in the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. If its centroid is the origin (0,0), find its area.

8. A sphere is inscribed inside a pyramid with a square as a base whose height is $\frac{\sqrt{15}}{2}$ times the length of one edge of the base. A cube is inscribed inside the sphere. What is the ratio of the volume of the pyramid to the volume of the cube?

9. How many hexagons are in the figure below with vertices on the given vertices? (Note that a hexagon need not be convex, and edges may cross!)



10. Let C_1 and C_2 be two concentric reflective hollow metal spheres of radius R and $R\sqrt{3}$ respectively. From a point P on the surface of C_2, a ray of light is emitted inward at 30° from the radial direction. The ray eventually returns to P. How many **total** reflections off of C_1 and C_2 does it take?

Team Test
Stanford Mathematics Tournament 2000

1. You are given a number, and round it to the nearest thousandth, round this result to nearest hundredth, and round this result to the nearest tenth. If the final result is .7, what is the smallest number you could have been given? As is customary, 5's are always rounded up. Give the answer as a decimal.

2. The price of a gold ring in a certain universe is proportional to the square of its purity and the cube of its diameter. The purity is inversely proportional to the square of the depth of the gold mine and directly proportional to the square of the price, while the diameter is determined so that it is proportional to the cube root of the price and also directly proportional to the depth of the mine. How does the price vary solely in terms of the depth of the gold mine?

3. Find the sum of all integers from 1 to 1000 inclusive which contain at least one 7 in their digits, i.e. find $7 + 17 + ... + 979 + 987 + 997$.

4. All arrangements of letters VNNWHTAAIE are listed in lexicographic (dictionary) order. If AAEHINNTVW is the first entry, what entry number is VANNAWHITE?

5. Given $\cos(\alpha+\beta) + \sin(\alpha-\beta) = 0$, $\tan\beta = \frac{1}{2000}$, find $\tan\alpha$.

6. If α is a root of $x^3 - x - 1 = 0$, compute the value of $\alpha^{10} + 2\alpha^8 - \alpha^7 - 3\alpha^6 - 3\alpha^5 + 4\alpha^4 + 2\alpha^3 - 4\alpha^4 - 6\alpha - 17$.

7. 8712 is an integral multiple of its reversal, 2178, as 8712=4*2178. Find another 4-digit number which is a non-trivial integral multiple of its reversal.

8. A woman has \$1.58 in pennies, nickels, dimes, quarters, half-dollars and silver dollars. If she has a different number of coins of each denomination, how many coins does she have?

9. Find all positive primes of the form $4x^4 + 1$, for x an integer.

10. How many times per day do at least two of the three hands on a clock coincide?

11. Find all polynomials $f(x)$ with integer coefficients such that the coefficients of both $f(x)$ and $[f(x)]^3$ lie in the set $\{0, 1, -1\}$

12. At a dance, Abhinav starts from point $(a, 0)$ and moves along the negative x direction with speed v_a, while Pei-Hsin starts from $(0, b)$ and glides in the negative y-direction with speed v_b. What is the distance of closest approach between the two?

13. Let $P_1, P_2, \ldots, P_n$ be a convex n-gon. If all lines P_iP_j are joined, what is the maximum possible number of intersections in terms of n obtained from strictly inside the polygon?

14. Define a sequence $<x_n>$ of real numbers by specifying an initial x_0 and by the recurrence $x_{n+1} = \frac{1+x_n}{1-x_n}$. Find x_n as a function of x_0 and n, in closed form. There may be multiple cases.

15. $\lim_{n\to\infty} nr \sqrt[2]{1 - \cos\frac{2\pi}{n}} = ?$