Advanced Topics Solutions
Stanford Mathematics Tournament 2000

1. Assume we have 4 colors - 1, 2, 3, and 4. Fix the bottom as color 1. On the remaining sides you can have colors 2, 3, 4 (in that order), or 2, 4, 3, which are not rotationally identical. So, there are **2** ways to color it.

2. Since $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, we can write the first term as $(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})^6$. Since $\cos\frac{4\pi}{3} = -\frac{1}{2}$ and $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, we can write the second term as $(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})^6$. Now, we apply DeMoivre's Theorem to simplify the first expression to $(\cos 6\cdot\frac{2\pi}{3} + \sin 6\cdot\frac{2\pi}{3}) = (\cos 4\pi + \sin 4\pi) = 1 + 0 = 1$. Similarly, we simplify the second expression to $(\cos 6\cdot\frac{4\pi}{3} + \sin 6\cdot\frac{4\pi}{3}) = (\cos 8\pi + \sin 8\pi) = 1 + 0 = 1$. Thus, the total sum is $1 + 1 = \mathbf{2}$.

3. We know that $\frac{1}{n^2+2n} = \frac{1}{n(n+2)} = \frac{\frac{1}{n} - \frac{1}{n+2}}{2}$. So, if we sum this from 1 to ∞, all terms except for $\frac{\frac{1}{1}}{2} + \frac{\frac{1}{2}}{2}$ will cancel out (a "telescoping" series). Therefore, the sum will be $\mathbf{\frac{3}{4}}$.

4. The possibilities for the numbers are:

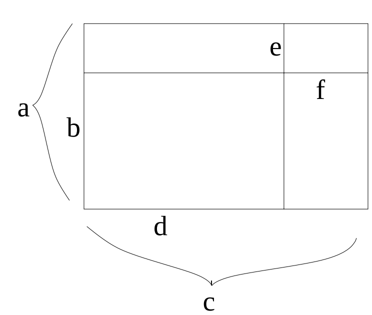
   - all five are divisible by 3
   - three are divisible by 3, one is $\equiv 1 \pmod 3$ and one is $\equiv 2 \pmod 3$
   - two are divisible by 3, and the other three are either $\equiv 1 \pmod 3$ or $\equiv 2 \pmod 3$
   - one is divisible by 3, two are $\equiv 1 \pmod 3$ and two are $\equiv 2 \pmod 3$
   - four are $\equiv 1 \pmod 3$ and one is $\equiv 2 \pmod 3$
   - four are $\equiv 2 \pmod 3$ and one is $\equiv 1 \pmod 3$

   This gives us 1001 possible combinations out of $\binom{15}{5}$ or 3003. So, the probability is $\frac{1001}{3003} = \mathbf{\frac{1}{3}}$.

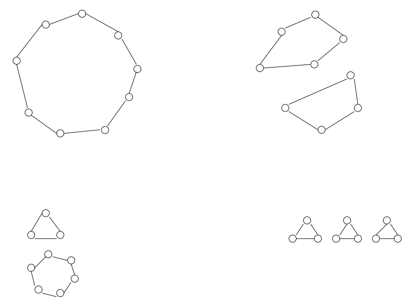
5. **153,370,371,407**

6. There are 6 people that could get their hat back, so we must multiply 6 by the number of ways that the other 5 people can arrange their hats such that no one gets his/her hat back. So, the number of ways this will happen is ($6\cdot$ derangement of 5), or $6 * 44 = 264$. Since there are $6! = 720$ possible arrangements of hats, the probability of exactly one person getting their hat back is $\frac{264}{720} = \mathbf{\frac{11}{30}}$.

7. We can view these conditions as a geometry diagram as seen below. So, we know that $\frac{e}{f} = \frac{3}{4}$ (since $e = a - b = \frac{3}{4}c - \frac{3}{4}d = \frac{3}{4}f$ and we know that $\sqrt{e^2+f^2} = 15$ (since this is $\sqrt{a^2+c^2} - \sqrt{b^2+d^2}$). Also, note that $ac + bd - ad - bc = (a-b)(c-d) = ef$. So, solving for e and f, we find that $e^2 + f^2 = 225$, so $16e^2 + 16f^2 = 3600$, so $(4e)^2 + (4f)^2 = 3600$, so $(3f)^2 + (4f)^2 = 3600$, so $f^2(3^2+4^2) = 3600$, so $25f^2 = 3600$, so $f^2 = 144$ and $f = 12$. Thus, $e = \frac{3}{4}\cdot 12 = 9$. Therefore, $\boldsymbol{ef} = 9 * 12 = \mathbf{108}$.



8. It suffices to consider the complements of the graphs, so we are looking for graphs with 9 vertices, where each vertex is connected to 2 others. There are **4** different graphs - see below.



9. The probability of the Reals hitting 0 singles is $(\frac{2}{3})^3$. The probability of the Reals hitting exactly 1 single is $\binom{3}{2}\cdot(\frac{2}{3})^3\cdot\frac{1}{3}$, since there are 3 spots to put the two outs (the last spot *m*ust be an out, since the inning has to end on an out). The probability of the Reals hitting exactly 2 singles is $\binom{4}{2}\cdot(\frac{2}{3})^3\cdot(\frac{1}{3})^3$. The probability of the Reals hitting exactly 3 singles is $\binom{5}{2}\cdot(\frac{2}{3})^3\cdot(\frac{1}{3})^3$. If any of these happen, the Alphas win right away. Adding these gives us a $\frac{656}{729}$ chance of this happening. If exactly 4 singles occur (with probability $\binom{6}{2}\cdot(\frac{2}{3})^3\cdot(\frac{1}{3})^4$), then there is a $\frac{2}{5}$ chance that the Alphas win. The probability of this happening is $\frac{2}{5}\cdot\frac{40}{729}$. , the total probability of the Alphas winning is the sum of these two probabilities, or $\frac{656}{729}+\frac{16}{729}=\mathbf{\frac{224}{243}}$.

10. A will say yes when B says no to $n-1$ or n, as A will then know B's number is one greater than A's number. Thus, A responds first, after $\boldsymbol{\frac{n-1}{2}}$ **"no" responses if $\boldsymbol{n}$ is odd**, after $\boldsymbol{\frac{n}{2}}$ **"no" responses if $\boldsymbol{n}$ is even**.

Algebra Solutions
Stanford Mathematics Tournament 2000

1. The only integers that satisfy $|x| + 5 < 7$ are the ones that satisfy $|x| < 2$ - namely, $-1, 0, 1$. The integers that satisfy $|x - 3| > 2$ are $6, 7, 8, \ldots$ and $0, -1, -2, \ldots$. So, the integers that satisfy both are $0, -1$, and there are **2** of them.

2. $2000^3 - 1999 \cdot 2000^2 - 1999^2 \cdot 2000 + 1999^3$ can be factored into $(2000 - 1999)2000^2 + 1999^2(-2000 + 1999)$, which reduces to $2000^2 - 1999^2$. This factors into $(2000 + 1999)(2000 - 1999)$, which is equal to **3999**.

3. Let the scores be a,b,c,d,e, where $0 \leq a \leq b \leq c \leq d \leq e \leq 100$. So, the mean is $\frac{1}{5}(a + b + c + d + e)$, and the median is c. So, we want to maximize $\frac{1}{5}(a + b + c + d + e) - c$. To do this, we must maximize d and e and minimize or maximize c. One way to do this is to let $a = b = c = 0$ and $d = e = 100$, so the difference between the mean and the median is $\frac{1}{5}(0 + 0 + 0 + 100 + 100) - 0) = \frac{200}{5} = \mathbf{40}$. If we maximize c, then $c = d = e = 100$, and then the mean is $\frac{1}{5}(0 + 0 + 100 + 100 + 100) = 60$, and the median is 60, with a difference of 40 as well.

4. A shortest path is $x \to x^2 \to x^4 \to x^8 \to x^{12} \to x^{24} \to x^{25} \to x^{50} \to x^{100} \to x^{200} \to x^{400} \to x^{800} \to x^{1600} \to x^{2000}$, using **13 multiplications**.

5. The price starts at \$100. Clearly, the order of price changes does not matter. It is reduced by 10% three times (\$100 $\to$ \$90 $\to$ \$81 $\to$ \$72.90), and the new price is \$72.90. It is increased by 10% four times (\$72.90 $\to$ \$80.19 $\to$ \$88.209 $\to$ \$97.0299 $\to$ \$106.73289) , and the new price is \$106.73289. Rounded to the nearest cent, this is **\$106.73**.

6. Every day Edward works, he gets $\frac{1}{9}$ of the test done. Similarly, every day Barbara works, she gets $\frac{1}{10}$ of the test done, every day Abhinav works, he gets $\frac{1}{11}$ of the test done, and every day Alex works, he gets $\frac{1}{12}$ of the test done. So, after 4 days (after everyone has worked on the test one day, they have completed $\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} = 38.535\%$ of the test. After 8 days, they have completed twice that, or 77.0707% of the test. After Edward, Barbara, and Abhinav each work one more day, the test will be complete in the minimum amount of time, so the test will take **11 days** to complete. If the least efficient workers work after the 8th day, the test still takes 11 days to complete.

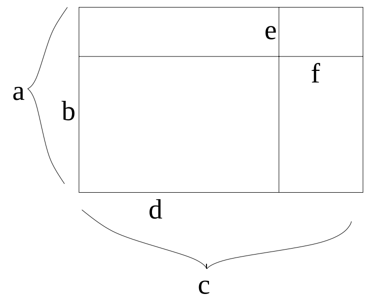
7. All multiplicatively perfect numbers have exactly 4 distinct positive divisors, or 1. So, we must look for numbers that are either

   - 1
   - a product of two distinct primes
   - a cube of a prime

   Numbers satisfying one of these conditions less than 100 are: 1, 6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87, 91, 93, 94, 95. There are **33** of these.

8. $168 = 2^3 \cdot 3 \cdot 7$. There are only 2 combinations of these whose sums allow indistinguishability of the ages. If there are 27 trees, 2, 4, 21 and 1, 12, 14 years are possible. If there are 21 trees, 2, 7, 12 and 3, 4, 14 are possible. So, the possible ages of the oldest daughter are **12, 14, 21**.

9. We can view these conditions as a geometry diagram as seen below. So, we know that $\frac{e}{f} = \frac{3}{4}$ (since $e = a - b = \frac{3}{4}c - \frac{3}{4}d = \frac{3}{4}f$ and we know that $\sqrt{e^2 + f^2} = 15$ (since this is $\sqrt{a^2 + c^2} - \sqrt{b^2 + d^2}$). Also, note that $ac + bd - ad - bc = (a - b)(c - d) = ef$. So, solving for e and f, we find that $e^2 + f^2 = 225$, so $16e^2 + 16f^2 = 3600$, so $(4e)^2 + (4f)^2 = 3600$, so $(3f)^2 + (4f)^2 = 3600$, so $f^2(3^2 + 4^2) = 3600$, so $25f^2 = 3600$, so $f^2 = 144$ and $f = 12$. Thus, $e = \frac{3}{4}12 = 9$. Therefore, $\boldsymbol{ef = 9 * 12 = 108}$.



10. $a = 1$ clearly does not work, since if $x = 1$, then $x^4 + a^2 = 2$, which is prime. $a = 2$ clearly does not work, since if $x = 1$, then $x^4 + a^2 = 5$, which is also prime. Here is a table for a's and values of x that show they do not work:

a	x	$a^4 + x^2$
3	10	10009
4	1	17
5	2	41
6	1	37
7	20	160049

So, consider $a = 8$; i.e. the sum x^4+64. This is the same as $(x^2+8)^2-16x^2=(x^2 + 4x + 8)(x^2 - 4x + 8)$ by the difference of squares. This is clearly not prime for any integer x. So, the answer is $\boldsymbol{a = 8}$.

Calculus Solutions
Stanford Mathematics Tournament 2000

1. $y = x^3 - 3x^2 + 6x + 2000$, so $y' = 3x^2 - 6x + 6$ and $y'' = 6x - 6$, so the point of inflection is the solution to $6x - 6 = 0$, or $x = 1$. At $x = 1$, the slope is $f'|_{x=1} = 3(1)^2 - 6(1) + 6 = \mathbf{3}$.

2. The change in Karen's position is $x - x^3$. The optimal length to climb is at a critical point. The only realistic critical point is at the solution to $1 - 3x^2 = 0$ or $\boldsymbol{x = \frac{\sqrt{3}}{3}}$.

3. $\int_0^{\frac{1}{4}\pi} \sin x + C = 0$, from the statement of the problem. So, $[-\cos x + Cx]|_0^{\frac{\pi}{4}} = 0$. Thus, $\cos\frac{\pi}{4} + \frac{\pi}{4}C + 1 = 0$. So, $-\frac{\sqrt{2}}{2} + \frac{\pi}{4}C + 1 = 0$, and solving for C, we find that $\boldsymbol{C = \frac{2\sqrt{2}-4}{\pi}}$.

4. Let $y = \tan x$. So, we want to find the minimum of $y + \frac{1}{y}$, where $0 \le y \le \infty$. Taking the derivative and minimizing, we find that the minimum occurs at $y = 1$, so the minimum of the given function occurs at $\arctan 1 = \boldsymbol{\frac{\pi}{4}}$.

5. $f(x) = \sum_{i=1}^{\infty} \frac{x^i}{i}$. So, $f'(x) = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$. Thus, $\boldsymbol{f(x) = -\ln(1-x)}$.

6. Assume the pipe barely fits around the corner (i.e. it is in contact with the corner). The lower corner is at $(0,0)$ and the upper corner is at $(6, 6\sqrt{5})$. Call x_0 the point on the lower wall it hits at the tightest spot. Given an x_0, the longest a pipe could be with one end at x_0 and leaning against the $(6, 6\sqrt{5})$ corner is $\sqrt{x_0^2 + \left(6\sqrt{5} + \frac{36\sqrt{5}}{x_0-6}\right)^2}$. We want the minimum of all of these "longest pipes", because the pipe needs to fit at all angles around the corner. Taking the derivative (without the square root for simplicity) and setting it equal to 0, we need to solve $x_0^3 - 6x_0^2 + 36x_0 - 1296 = 0$. We can quickly find that $x_0 = 12$ is the only good solution, so the maximum length is $\mathbf{12\sqrt{6}}$.

7. For each value of x, we want to find the minimum (maximum) of y for the range of x_0. Therefore, take $\frac{dy}{dx_0}$, treating x as a constant. Set this equal to 0, and solve for x_0 relative to x. Plug this in for x_0 in the given family to obtain the envelope $\boldsymbol{y = x + \frac{1}{x}, x \neq 0}$.

8. Let I denote the given integral. Under the transformation $\theta \to \frac{\pi}{2} - \theta$, I transforms to $\int_0^{\frac{\pi}{2}} \ln(\cos(\theta))d\theta$. So,

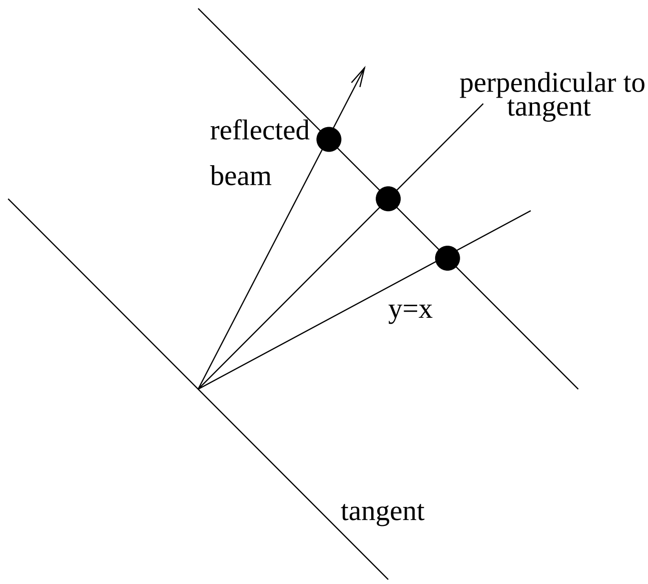
$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \ln(\sin\theta\cos\theta)d\theta \\ &= \int_0^{\pi} (\ln(\sin 2\theta) - \ln 2)d(2\theta)/2 \\ &= -\frac{\pi}{2\ln 2} + \frac{1}{2}\int_0^{\pi} \sin(\alpha)d\alpha \\ &= -\frac{\pi}{2}\cdot\ln 2 + \int_0^{\frac{\pi}{2}} \sin(\alpha)d\alpha \\ &= -\frac{\pi}{2}\cdot\ln 2 + I \end{aligned}$$

giving $\boldsymbol{I = -\frac{\pi}{2}\ln 2}$.

9. Note first that $\left([f(x)]^2 - x\right)^2 = f(x)$, so if $f(a) = 4$, then $(16-a)^2 = 4$, so $a = 14$. Now, $f'(x) = \frac{1 + \frac{f'(x)}{2([f(x)]^2 - x)}}{2f(x)}$, so $f'(14) = \mathbf{\frac{4}{31}}$.

10. Solution: Throughout this solution we will use the fact that when light bounces off a mirror, the angle of incidence is equal to the angle of reflection. First the beam hits the point (8,-1), then (6,1), (4,-1), (2,1), and then is travelling along the line $y = x - 1$. Thus the beam hits the parabola at the point $(1 + \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$. To estimate $\sqrt{5}$, notice that $22^2 = 484$ and $23^2 = 529$, so $\sqrt{5} = \frac{\sqrt{500}}{10} = 2.2\ldots$. Thus $\frac{1-\sqrt{5}}{2} = -.6\ldots$, so the light hits the parabola at approximately (.4,-.6). The slope of the tangent to the parabola at this point is $\frac{-1}{2}(.4)^{-1/2}$, which is about -.8, so we need to find the slope of the beam after it reflects off of this tangent. For purposes of finding this slope, change coordinates so that the point of intersection is the origin. The beam is coming in along $y = x$, and $y = 1.2x$ is perpendicular to the tangent. The diagram below should clarify the setup.

    We will find the new path of the light by finding the reflection about the line $y = 1.2x$ of a point on its incoming path. We know the point (1,1.2) is on the line $y = 1.2x$, so a perpendicular through this point is $y - 1.2 = -.8(x-1)$, which intersects $y = x$ at the point (1.1,1.1). Thus the new path goes through the point (.9,1.3), so it has slope 1.4 (all values rounded to one decimal place). Going back



to our original coordinate system, the light is now travelling along the line $y + .6 = 1.4(x - .4)$, so it next hits the mirror at (1.5,1). After that the x coordinate increases by $2/1.4 = 1.4$ between bounces, so it hits (2.9,-1), (4.3,1), (5.7,-1), (7.1,1), (8.5,-1), and finally (9.9,1). A closer examination of the approximations made (e.g. by refining them to two decimal places) reveals that the last bounce is actually further to the left (at (9.21,1), to be more precise), so indeed the light does bounce **12** times.

General Solutions
Stanford Mathematics Tournament 2000

1. Since $d = a - c$, substitute into the equation $b = 2c + d$ to get $b = 2c + a - c = a + c$. Also, substitute into $2c = d + a - 1$ to get $2c = a - c + a - 1$, or $3c = 2a - 1$. Now, since $b = a + c$, we can substitute into $a = 2b + c$ to get $a = 2a + 2c + c$, or $a = -3c$. Since we know $3c = 2a - 1$ from above, we substitute in to get $3c = -6c - 1$, or $c = -\frac{1}{9}$. Thus, we find that $a = \frac{1}{3}$, $d = \frac{4}{9}$, and $\boldsymbol{b = \frac{2}{9}}$.

2. Let x be the temperature we are looking for, so $x = \frac{9}{5} + 32$. So, $-\frac{4}{5}x = 32$, so $x = -\frac{5}{4} \cdot 32 = \mathbf{-40}$.

3. Let x be the length of Henry's shadow in feet. Using similar triangles, we find that $\frac{5.5}{12} = \frac{x}{5}$, so $x = \frac{5.5}{12} \cdot 5 = \frac{11}{24} \cdot 5 = \mathbf{\frac{55}{24}}$.

4. Let x be the number of students and y be the number of non-students. We then have the equations $x + y = 3000$ and $10x + 15y = 36250$. Substituting, we find that $10(3000 - y) + 15y = 36250$, or $30000 - 10y + 15y = 36250$, so $30000 + 5y = 36250$. So, $5y = 6250$, and $y = 1250$, so $x = \mathbf{1750}$.

5. The total number of degrees in an octagon is $(8 - 2) \cdot 180 = 1080$. Since the degrees are evenly distributed among the angles, the measure of one interior angle is $\frac{1080}{8} = \mathbf{135^\circ}$.

6. Pick any card first, then pick the other face-down card.

   (a) $\mathbf{\frac{1}{3}}$

   (b) $\mathbf{\frac{2}{3}}$

7. $\sqrt{19992000} \approx 4471.241$, and $[4471.241] = \mathbf{4471}$.

8. The time that Bobo can juggle is the number of cows times seconds. So, we get the following table:

cows started juggling	1	2	3	4	5	6	7	8	9	10	11
total time	64	110	141	160	165	162	154	144	126	130	132
cows started juggling	12	13	14	15	16	17	18	19	20	21	22
total time	132	130	126	120	112	102	90	76	60	42	22

   Thus, we see that the maximum occurs with **5 cows**, and the total time is 165 seconds = $\mathbf{2\frac{3}{4}}$ **minutes**.

9. Let p be the price of HMMT. So $p = k \cdot \frac{x}{y}$, where k is a constant to be determined. We know that when $x = 8$ and $y = 4$ that $p = 12$, so, solving for k, we find that $k = 6$. So, when $x = 4$ and $y = 8$, we find that $p = 6 \cdot \frac{4}{8} = \mathbf{3}$.

10. On the 12 foot sides, he needs 7 posts, and on the 20 foot sides, he needs 9 posts, so he needs $7 + 9 + 7 + 9 = 32$ total posts. Let x be the number of normal fenceposts and y be the number of strong fenceposts, so $x + y = 32$. To spend \$70, we have the equation $2x + 3y = 70$. Substituting, we find that $2(32 - y) + 3y = 70$, so $64 - 2y + 3y = 70$, and $64 + y = 70$, so $y = \mathbf{6}$.

11. Substituting, we find that $\frac{3+n}{3-n} = 3$, so $3 + n = 9 - 3n$, thus $4n = 6$. So, $n = \mathbf{\frac{3}{2}}$.

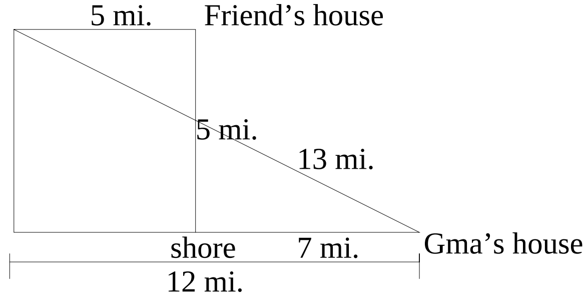
12. Yes - one possible path is $4 \to 1 \to 2 \to 4 \to 5 \to 2 \to 3 \to 5$, so the difference between the start island and end island is **1**.

13. The number of rearrangements keeping 1 number in its spot and rearranging the other 5 such that none are in the right spot is 44. There are 6 numbers to fix, this gives us an answer of $44 \cdot 6 = \mathbf{264}$.

14. Feburary 26, 2000 is a Saturday. April 24, 2000 is $23 \cdot 365 + 1 + 1 + 1 + 1 + 1 + 1 = 8401$ days away from April 24, 1977 (including the leap years). So, Feburary 26, 2000 is $8401 - 3 - 31 - 24 = 8343$ days after April 24, 1977. Now, $\frac{8343}{7} = 1191$ with a remainder of 6. So, 6 days before Saturday is **Sunday**.

15. Notice that $3^6 = 729$ while $5^4 = 625$, and since $\ln 5 > \ln 3$, it follows that $5^4 \ln 3 < 3^6 \ln 5$, so $5^3 * 5 \ln 3 < 3^5 * 3 \ln 5$, and and so by laws of logarithms, $5^3 \ln 3^5 < 3^5 \ln 5^3$. Again applying laws of logarithms, it follows that $\ln (3^5)^{(5^3)} < \ln (5^3)^{(3^5)}$. So, since $\ln x$ is an increasing function, it follows that $(3^5)^{(5^3)} < (5^3)^{(3^5)}$, and so it follows that $\mathbf{(5^3)^{(3^5)}}$ **is greater**. One can also use the laws of exponents to reduce the values to $3^{(5^4)}$ and $5^{(3^6)}$. The second is clearly larger.

16. Expressing the total time Joe has biked in hours leads to the equation $\frac{x}{20} + \frac{x}{20} + \frac{x+2}{14} = 1$. So, $x = 5$. Thus, we can construct the diagram below, and find the total time that it takes to get back: $13 * \frac{1}{78} = \frac{1}{6}$ hours, or **10 minutes**.



17. There are 7! ways to arrange those letters. However, for every distinct arrangement, there are $2! * 2! = 4$ total arrangements of the 2 T's and 2 N's. Therefore, the total number of distinct ways to arrange the letters is $\mathbf{\frac{7!}{2!2!} = 1260}$.

18. The only digits possible are 4, 6, 8, and 9. The only groups of numbers allowed keeping the average at 5 are 8444 are 6464. There are 4 ways to arrange 8444 and 6 ways to arrange 6464 so there are only **10 combinations** to try.

19. Let y be the number of coins in the chest. From the problem, we know that $y \equiv 5 \pmod{11}$, $y \equiv 3 \pmod{10}$, and $y \equiv 0 \pmod{9}$. Combining these gives us that $y \equiv 423 \pmod{990}$, so the answer is **423**.

20. The area of the big circle is $\left(\frac{5}{2}\right)^2 \pi = \frac{25}{4}\pi$. The area of the circle with diameter AB is π, and the area of the circle with diameter BC is $\frac{9}{4}\pi$. Thus, the percentage of the big circle that is shaded is $\frac{\frac{25}{4}\pi - \pi - \frac{9}{4}\pi}{\frac{25}{4}\pi} = \frac{25-4-9}{25} = \mathbf{\frac{12}{25} = 48\%}$.

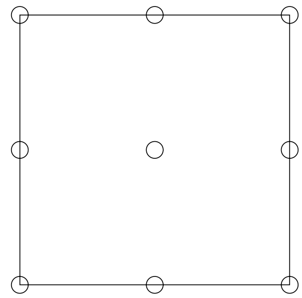
21. It is clear that we can split the figure into 12 equilateral triangles, all of which have side length s. So, since the area of one of these triangles is $\frac{s^2\sqrt{3}}{4}$, the total area is $\mathbf{3s^2\sqrt{3}}$.

22. $A_{iso} = \frac{A_{eq}}{2}$ so $\Delta A = A_{eq} - A_{iso} = \frac{1}{2}A_{eq}$. Also, $A_{eq} = \frac{4^2 \cdot \sqrt{3}}{4} = 4\sqrt{3}$, so $\Delta A = 2\sqrt{3}$. Thus, the area is $\mathbf{2\sqrt{3}}$.

23. Notice that $7^{2k+1} \equiv 3 \pmod{4}$ for $k \in \mathbf{Z}$ (k is an integer). Also, $7^1 \equiv 7 \pmod{100}$, $7^2 \equiv 49 \pmod{100}$, $7^3 \equiv 43 \pmod{100}$, $7^4 \equiv 1 \pmod{100}$, with the cycle repeating afterwards. So, clearly $7^{7^{7^7}} \equiv 3 \pmod{4}$, and since the cycle has period $4 \pmod{100}$, we can conclude that $\mathbf{7^{7^{7^7}} \equiv 43 \pmod{100}}$.

24. If there are no empty boxes, there are 126 ways of distributing the identical candy. If there is one empty box, there are 84 ways of distributing the candy and 5 ways of choosing the empty box, so there are $84 \cdot 5 = 420$ ways. If there are two empty boxes, there are 36 ways of distributing the candy and 6 combinations of assigning the empty boxes so that they are not adjacent, so there are $36 \cdot 6 = 216$ ways. If there are three empty boxes, there are 9 ways of distributing the candy, and only 1 way of arranging the empty boxes. Having four or five empty boxes is impossible, since some two would have to be adjacent. So, the total number of ways is $126 + 420 + 216 + 9 = \mathbf{771}$.

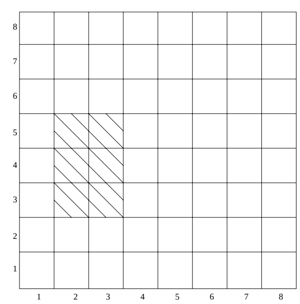
25. We can place 9 points as shown, all at least $\frac{1}{2}$ unit apart, but the next point must be less than $\frac{1}{2}$, so **10** points must be placed. There is no arrangement of 10 points with distance at least $\frac{1}{2}$. The proof of this is a simple application of the Pigeonhole Principle.



26. Janet is at $(5, -5)$ and Tim is at $(7, 4)$. They are $\boldsymbol{\sqrt{85}}$ miles apart.

Geometry Solutions
Stanford Mathematics Tournament 2000

1. Consider the board labeled as below, with labels for columns and rows. To choose any rectangle on the board, it is sufficient to choose some number (1-8) of adjacent columns, and some number (1-8) of adjacent rows, since the rectangle can be created by forming the intersection of the columns and rows. For instance, the intersection of columns 2,3 and rows 3,4,5 is the rectangle shaded below. So, there are 8 ways to choose 1 adjacent column, 7 ways to choose 2 adjacent columns, $\ldots$, 1 way to choose 8 adjacent columns, so there are $8+7+6+5+4+3+2+1 = 36$ total ways to choose the columns, and 36 ways to choose the rows. Thus, the total number of ways to choose a rectangle (i.e. the total number of rectangles) is $\mathbf{36^2 = 1296}$.

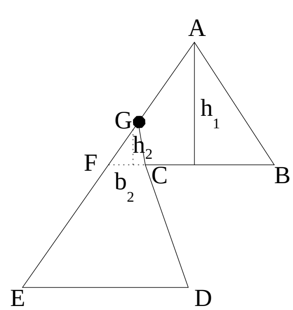


2. The maximum occurs in an equilateral triangle, in which case the sides $a = b - c$ are given by $a^2 + b^2 + c^2 = 3a^2 = 96$, so $a = 4\sqrt{2}$. Thus, the medians are $3 \cdot (\frac{a\sqrt{3}}{2}) = \mathbf{6\sqrt{6}}$.

3. Since $ABCD$ is a square, we can write $(PA)^2 + (PC)^2 = (PB)^2 + (PC)^2$. So, we can substitute in for PA,PC, and PD to get that $PB = \mathbf{24}$.

4. Notice that in general, when there is a rectangle of side length x and y, the area of the non-triangle regions (created by drawing a line connecting the midpoint of two opposite lines and a line connecting two opposite corners - see diagram for examples) is simply $\frac{3}{4}$ of the original area of the box, since the area of the excluded triangles are $\frac{1}{2} \cdot \frac{y}{2} \cdot \frac{x}{2} + \frac{1}{2} \cdot \frac{y}{2} \cdot \frac{x}{2} = 2 \cdot \frac{xy}{2} = \frac{xy}{4}$, so the desired area is simply $ab - \frac{ab}{4} = \frac{3}{4}ab$, as desired. So, if we consider the four rectangles making up the large rectangle that are divided this way (the one with dimensions $\frac{a}{4}$x$\frac{b}{3}$, etc.), we can say that the total shaded area is $\frac{3}{4}$ of the total area of the rectangle - that is, $\boldsymbol{\frac{3}{4}ab}$.

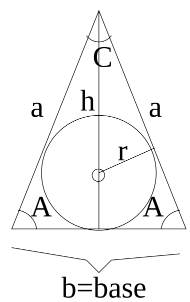
5. The area of $\triangle ABF = \frac{1}{2}bh = \frac{1}{2} \cdot 3h = \frac{3}{2}(3\sin 60°) = \frac{9}{2}\sin 60°$. The area of $\triangle FCDE =$ area of $\triangle ABF$ - area of $\triangle FGC = \frac{9}{2}\sin 60° - \frac{1}{2}\sin 60° = 4\sin 60°$. Therefore, area of $ABCDE =$ area of $\triangle ABF +$ area of $FCDE = \frac{9}{2}\sin 60° + 4\sin 60° = \frac{17}{2}\sin 60° = \mathbf{\frac{17\sqrt{3}}{4}}$.



6. The area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$. Therefore, the side length is $4\sqrt{2}$ and height $h = 2\sqrt{6}$. Now, if r is the radius of the inscribed circle, then $r = \frac{h}{3}$, since we have an equilateral triangle. Thus the area is $\pi r^2 = \mathbf{\frac{8\pi}{3}}$.

7. Let's "resize" the coordinates to be $x' = x, y' = \frac{2y}{3}$. This keeps the origin at (0,0), but turns our ellipse into a circle of radius 2. Thus now, the triangle is equilateral, and we can see it now has area $3\sqrt{3}$. Once we expand back, we can see we are just multiplying the area by $\frac{3}{2}$ and so the answer is $\mathbf{\frac{9\sqrt{3}}{2}}$.

8. By standard formula, we have that the radius of the inscibed circle, r, is $r = \frac{ab\sin A}{2a+b}$ (isoceles triangle that is formed by cutting the pyramid vertically in half (cuts the base into 2 equal rectangles)). $h^2 + \left(\frac{b}{2}\right)^2 = a^2$ gives $a = \sqrt{h^2 + \frac{b^2}{4}}$. Also $\sin A = \frac{h}{a}$. Therefore, $r = \frac{bh}{2\sqrt{h^2+\frac{b^2}{4}}+b}$. Note that the diameter of the cube is the diameter of the sphere. Let l be the length of the side of the cube, so the diameter of the cube is $l\sqrt{3} = 2r$, so $l = \frac{2r}{\sqrt{3}}$. So, the volume of the pyramid is $\frac{1}{3}b^2h$ and the cube volume is l^3. So, the ratio is $\mathbf{\frac{25\sqrt{3}}{6}}$.



9. A hexagon can be formed by removing any vertex, removing all vertices connected to that vertex, and then removing any edges connected to any of the removed vertices, and these are the only hexagons in the diagram. Thus, since there are 10 vertices, there are **10 hexagons** in the figure.

10. Consider the flattened version of the situation. Then let O be the center fo the spheres, A be the first reflection point, B be the point of C_1 such that OB is perpendicular to OP. Then since $OB = 1, OP = \sqrt{3}$, $\angle PBO = 60°$ and P, B, A are collinear, $\angle BAO = 60°$ implies that $\angle POA = 30°$. Therefore, each reflection takes the ray $\frac{1}{12}$ around the circle, so there are **11** reflections.

Team Test Solutions
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1. .6445 rounds to .645 to .65 to .7. Otherwise .6444... rounds to .644. So the smallest number is **.6445**.

2. Let c =price, p=purity, d =diameter, h=depth of gold mine, k_i=constant. We are given $c = k_1 p^2 d^3$, $p = k_2 \frac{c^2}{h^2}$, and $d = k_3 \sqrt[3]{c}h$. So, $c = k_1 k_2^2 \frac{c^4}{h^4} k_3^3 ch^3 = k_4 c^5 \frac{1}{h}$. Thus, $k_4 c^4 = h$, and $c = k_5 h^{\frac{1}{4}}$. Thus, p varies as $\boldsymbol{h^{\frac{1}{4}}}$.

3. The sum of the numbers from 700 to 799 is $\frac{799 \cdot 800}{2} - \frac{699 \cdot 700}{2} = 74950$. The sum of the numbers from 70 to 79 is $\frac{79 \cdot 80}{2} - \frac{69 \cdot 70}{2} = 745$. So, all numbers that end from 70 to 79 (excluding those starting with 7, since we counted those from 700 to 799) is $745 \cdot 9 + 10(100 + 200 + \ldots + 600 + 800 + 900) = 44705$. The sum of all numbers ending in 7 is $9(7+17+27+37+47+57+67+87+97)+9(100+200+\ldots+600+800+900) = 38187$. So, the total sum of numbers containing a 7 is $74950 + 44705 + 38186 = \mathbf{157842}$.

4. **738,826**. This can be arrived at by stepping down, starting with finding how many combinations are there that begin with a letter other than V or W, and so forth. The answer is $\frac{8 \cdot 9!}{2 \cdot 2} + \frac{4 \cdot 7!}{2} + 4 \cdot 6! + 4 \cdot 4! + 3! + 2! + 2! = \mathbf{738826}$.

5. $0 = \cos(\alpha + \beta) + \sin(\alpha - \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta + \sin\alpha \cdot \cos\beta - \sin\beta\cos\alpha = (\cos\alpha + \sin\alpha) \cdot (\cos\beta - \sin\beta)$. So $\cos\alpha + \sin\alpha = 0$ or $\cos\beta - \sin\beta = 0$. Then $\tan\alpha = -1$ or $\tan\beta = 1$. Since $\tan\beta$ is given as $\frac{1}{2000}$, $\tan\alpha = \mathbf{-1}$.

6. Since $\alpha^3 - \alpha - 1 = 0$, then $\alpha^{10} = \alpha^8 + \alpha^7$. So, we can reduce our expression to $3\alpha^8 - 3\alpha^6 - 3\alpha^5 + 4\alpha^4 + 2\alpha^3 - 4\alpha^2 - 6\alpha - 17$. Also, $3\alpha^8 - 3\alpha^6 - 3\alpha^5 = 0$, so our expression reduces to $4\alpha^4 + 2\alpha^3 - 4\alpha^2 - 6\alpha - 17$. Also, $4\alpha^4 - 4\alpha^2 - 4\alpha^0$, so our expression reduces to $2\alpha^3 - 2\alpha - 17$. Now, $2\alpha^3 - 2\alpha - 2 = 0$, so our expression reduces to **-15**, which is our answer.

7. Another 4-digit number that satisfies this property is **9801**, since 9801=9*1089.

8. If she has a silver dollar, then she would have too many other coins, as 0 half dollars, 2 quarters, 3 dimes, etc. would be greater than the total. So she has no silver dollars, and at least one of every other denomination. Continuing, it turns out the only feasible solution is 0 silver dollars, 1 half dollar, 2 quarters, 3 dimes, 4 nickels, 8 pennies, for a total of **18** coins.

9. It suffices to consider $x \geq 1$, since $4(-x)^4 + 1 = 4(x)^4 + 1$, and $4(0) + 1 = 1$ is not prime. So, $4x^4 + 1 = (4x^4 + 4x^2 + 1) - 4x^2 = (2x^2 + 1)^2 - (2x)^2 = (2x^2 + 1 - 2x)(2x^2 + 1 + 2x)$. For integers x, both $2x^2 - 2x + 1$ and $2x^2 + 2x + 1$ are integers, so this factors $4x^4 + 1$ unless $2x^2 - 2x + 1 = \pm 1$ or $2x^2 + 2x + 1 = \pm 1$. Since $x > 0$, then $2x^2 + 2x + 1 > 1$, so we must have $2x^2 - 2x + 1 = \pm 1$. $2x^2 - 2x + 1 = -1$ is absurd ($4x^4 + 1, 2x^2 + 2x + 1 > 0$, so $2x^2 - 2x + 1 = \frac{4x^4+1}{2x^2+2x+1} > 0$), so we solve $2x^2 - 2x + 1 = 1$, or $2x^2 - 2x = 0$, so $x(x-1) = 0$, and $x = 0$ or $x = 1$. We have already rejected $x = 0$, so the only case left is $x = 1$, or $4(1)^4 + 1 = \mathbf{5}$.

10. The second hand crosses the minute hand 59 times an hour. The second hand crosses the hour hand 60 times an hour, except for 2 of the hours, due to the movement of the hour hand. The minute hand and the hour hand cross 22 times total, because the hour hand completes 2 rotations in a day, and the minute hand completes 24. The second, hour, and minute hand all coincide only at noon and midnight, but we've counted each of these 12:00's 3 times instead of once. Therefore, the answer is $59 \cdot 60 + 60 \cdot 24 - 2 + 22 - 2 \cdot 2$, giving us **2872** crossings.

11. $f(x)$ is either **0** or something of the form $\boldsymbol{\pm x^m}$, where $m \geq 0$.

12. A's position is $(a - V_a t, 0)$ and P's position is $(0, b - V_b t)$. So, at time t, the distance between them is $\sqrt{(a - V_a t)^2 + (b - V_b t)^2}$. Notice that this distance is the same as the distance between the point (a, b) and the line $(V_a t, V_b t)$, which is the same as the line $V_b x - V_a y = 0$. The distance from a line $Ax + By + C = 0$ and (x_0, y_0) is $\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, so the answer is $\boldsymbol{\frac{|bV_a - aV_b|}{\sqrt{V_a^2 + V_b^2}}}$

13. Given any 4 vertices, there is exactly one intersection of all the diagonals connecting them. So, the answer is $\boldsymbol{\binom{n}{4}}$.

14. x_0 if $n \equiv 0 \pmod 4$,$\frac{1+x_0}{1-x_0}$ if $n \equiv 1 \pmod 4$, $-\frac{1}{x_0}$ if $n \equiv 2 \pmod 4$, $\frac{x_0-1}{x_0+1}$ if $n \equiv 3 \pmod 4$.

15. Consider a regular n-gon with radius r. Let x be the side length of the n-gon. So, since the central angle is $\frac{2\pi}{n}$ (see diagram below), use the Law of Cosines to find that $x^2 = r^2 + r^2 - 2r * r \cos \frac{2\pi}{n}$, so $x^2 = 2r^2(1 - \cos \frac{2\pi}{n})$. Thus, $x = r\sqrt{2}\sqrt{1 - \cos \frac{2\pi}{n}}$. So, the total perimeter of the n-gon is $nx = nr\sqrt{2}\sqrt{1 - \cos \frac{2\pi}{n}}$. Now, if we take $\lim_{n\to\infty}$ of the perimeter, the result will be $2\pi n$, since the n-gon approaches a cirle, so $\lim_{n\to\infty} nr\sqrt{2}\sqrt{1 - \cos \frac{2\pi}{n}} = 2\pi r$, and so $\boldsymbol{\lim_{n\to\infty} nr\sqrt{1 - \cos \frac{2\pi}{n}} = \pi r\sqrt{2}}$.